Mathematics

for the IB Middle Years Programme

MYP Year 3

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Contents

Course Structure

Mathematics can be fun!

'I can't do it!'

Have you ever thought or exclaimed these words when stumped by a mathematics problem? I bet every one of us has said these words to themselves at least once in their lifetime. And not just because of a mathematics problem! In order to be engaged as a learner, regardless of age, we like to experience things in a fun and interactive way. Not only that, learning only happens when we leave our comfort zone.

This series is dedicated to the idea that mathematics can be (and is) fun.

Our mission is to accompany you, dear learner, out of your comfort zone and towards the joy of mathematics. Do you accept this challenge?

Bearing in mind the latest research about learning mathematics, the driving ideas behind this series are the following:

- We believe that everyone can do mathematics. Of course there are a few that find it 'easier' than others, *but mathematics learning*, done right, *is for all*. We believe in you, the learner.
- *The essence of mathematics is solving problems*. We will work together to help you become a better problem solver. Problem solvers make mistakes. Plenty of them. With perseverance, they end up solving their problems. You can too. Making mistakes and learning from them is part of our education. Our approach is backed by research on *growth mindsets* and follows in the steps of George Pólya, the father of problem solving.
- Other than some special inventions, most of our societies' development is done by groups. That is why we will, with the help of your teachers, support you to achieve your goals within a group environment.
- Mathematicians' work, no matter how 'advanced' the result, starts with an exploration. Ideas do not magically materialise to a mathematician's mind by superpowers. Mathematicians work hard, and while working, discoveries are made. Whoever discovered gravitational forces did not sit back and then all of a sudden come out with the idea. It was observation first.
- Once you have an idea, you can investigate it to develop your understanding in more depth. We have included many opportunities for you to expand your knowledge further.

Fact

The legend is that Isaac Newton discovered gravity when he saw a falling apple while thinking about the forces of nature. Whatever really happened, Newton realised that some force must be acting on falling objects like apples because otherwise they would not start moving from rest.

It is also claimed that Indian mathematician and astronomer Brahmagupta-II (598–670) discovered the law of gravity over 1000 years before Newton (1642–1727) did. Others claim that Galileo discovered it 100 years before Newton.

How to use this book

No one can teach you unless you want to learn. We believe that, through this partnership with you, we can achieve our goals.

In this book, we have introduced each concept with an Explore. First and foremost, when you start a new concept, try to do the Explore. Have courage to make guesses but try to justify your guesses. Remember, it is ok to make mistakes. Work with others to analyse your mistakes.

Explore 9.4

Look back at the data for rolling a dice 36 times from Explore 9.3.
How would you represent this data to make it easy to read?

Throughout this book, you will find worked examples. When you are given a worked example, do not jump immediately to the solution offered. Try it yourself first. When you do look at the solution offered, be critical and ask yourself: could I have done it differently?

Worked example 8.3

Marta wants to cut 1 m of ribbon into three equal lengths.

How long should each length be?

Give your answer to a suitable degree of accuracy.

Solution

$100 \div 3 = 33.333\ldots$ cm

It is not possible to measure 33.333… cm accurately.

Each length is 33.3 cm (to the nearest mm).

At the end of any activity, we encourage you to reflect on what you have done. There is always a chance to extend what you have learned to new ideas or different perspectives. Not only in studying mathematics, but in any task you should always take the opportunity to reflect on what you did. You will either feel that the task is completed, or you may find that you need to improve on some parts of it. This is true whether you are a student, a teacher, a parent, an engineer, or a business leader, to mention a few. You will find reflection boxes throughout the book to help you with this.

Reflect

In Worked example 8.2, how did it help to convert the measurements to cm?

Can you round 5.26 m to the nearest 10 cm without converting to cm first?

At the end of each section of the book, you will find practice questions. It is recommended that you do these, and more, until you feel confident that you have mastered the concept at hand.

Practice questions 8.2

1 Write 7.517 m as ___ m ___ cm ____ mm

2 Write 3 m 24 cm 5 mm:

 a in cm, to the nearest mm

 b in m, to the nearest mm.

Instead of summarising each chapter for you, we have you review what you learned from the chapter in a self-assessment. These self-assessments are checklists. Look at them, and if you feel you missed something, revisit the section covering it.

Self assessment

- ☐ I can identify natural numbers, integers and real numbers.
- ☐ I can identify and use the place value of digits in natural numbers up to hundreds of millions.
- ☐ I can identify and use the place value of digits in decimals.

Finally, at the end of every chapter, it is good practice to look back at the chapter as a whole and see whether you can solve problems. Each chapter contains check your knowledge questions for this purpose.

Check your knowledge questions

1 Complete these measurement conversions.

a $420\,\text{m} = \square\,\text{cm}$ b $530\,\text{cm} = \square\,\text{m}$

c $0.4\,\text{km} = \square\,\text{mm}$ d $546\,\text{mm} = \square\,\text{cm}$

e $2450\,\text{m} = \square\,\text{km}$ f $3.4\,\text{m} = \square\,\text{mm}$

2 How many books of width 18 mm will fit on a shelf 1 m long?

During your course, your teacher will help you work in groups. In group work, ask for help and help others when asked. The best way of understanding an idea is when you explain it to someone else.

Remember, mathematics is not a bunch of calculations. Mathematical concepts must be communicated clearly to others. Whenever you are performing a task, justify your work and communicate it clearly.

Additional features

Matched to the latest MYP Mathematics Subject Guide

Key concepts, related concepts and global contexts

Each chapter covers one key concept and one or more related concepts in addition to being set within a global context to help you understand how mathematics is applied in our daily lives.

KEY CONCEPT

Relationships

RELATED CONCEPTS

Patterns, Quantity, Representation, Systems

GLOBAL CONTEXT

Globalisation and sustainability

Statement of inquiry and inquiry questions

Each chapter has a statement of inquiry and inquiry questions that lead to the exploration of concepts. The inquiry questions are categorised as factual, conceptual and debatable.

Statement of inquiry

Using number systems allows us to understand relationships that describe our climate, so we are able to acknowledge human impact on global climate change.

Approaches to learning tags

We have identified activities and questions that have a strong link to specific approaches to learning to help you understand where you are using particular skills.

Thinking skills

Do you recall?

At the start of each chapter, you will find do you recall questions to remind you of the relevant prior learning before you start a new chapter. Answers to the do you recall questions can be found in the answers section at the back of the book.

Do you recall?

1 What are directed numbers?

2 What are the number operations?

3 What mental methods do you know for adding two 2-digit numbers?

4 What mental methods do you know for subtracting from a 2-digit number?

Investigations

Throughout the book, you will find investigation boxes. These investigations will encourage you to seek knowledge and develop your skills. They will often provide an opportunity for you to work with others.

Investigation 1.1

Collect magazine, newspaper or online articles that use global temperatures, sea levels and carbon dioxide emissions. Explain in each case what the data tells you.

Research temperatures and the amount of rainfall for five different locations on the same day or month each year for 20 years. What does your data show you?

Fact boxes

Fact boxes introduce historical or background information for interest and context.

Fact

A pescatarian is someone who eats fish, but does not eat any other meat.

Hint boxes

Hint boxes provide tips and suggestions for how to answer a question.

Hint Q9

Note the different units.

Reminders

These boxes are used to recap previous concepts or ideas in case you need a refresher.

Reminder

Always state how you have rounded the measurement in the final answer.

Connections

These boxes highlight connections to other areas of mathematics, or even other subjects.

Connections

You learned how to measure and draw angles accurately in Chapter 4.

Challenge Q12

Challenge tags

We have identified challenging questions that will help you stretch your understanding.

This series has been written with inquiry and exploration at its heart. We aim to inspire your imagination and see the power of mathematics through your eyes.

We wish you courage and determination in your quest to solve problems along your your MYP mathematics journey. Challenge accepted.

Ibrahim Wazir, Series Editor

A note for teachers

Alongside the textbook series, we have also created digital Teacher Guides. These Guides include, amongst other things, ideas for group work, suggestions for organising class discussion using the Explores and detailed, customisable unit plans.

1 Year 2 review

1 Year 2 review

KEY CONCEPT

Relationships

RELATED CONCEPTS

Equivalence, Representation, Systems

GLOBAL CONTEXT

Scientific and technical innovation

Statement of inquiry

Systems and relationships can help us to represent scientific and technological developments in an advancing world.

Factual

- What determines if a number is a prime number?
- How do you perform operations on algebraic expressions?

Conceptual

- Why do we need an order of operations?
- How do you find the area of a composite object?

Debatable

- Which central value is more useful: mean or median?
- Can we say: I am more than 100% sure?

Do you recall?

1 Consider the set of numbers $3, -5, 3.2, 2.333\ldots, \sqrt{2}, \frac{3}{5}, 2^{-3}$

Write down the numbers that are:

- integers
- rational numbers
- irrational numbers
- natural numbers
- real numbers

2 Is 24 divisible by 3 and 5? Explain your thinking.

3 What is the result of $24 \div 8 \times 3$? Is it 1 or 9? Explain your answer.

1.1 Numbers

1.1.1 Classifying numbers

Explore 1.1

Classify these numbers as natural numbers, integers, rational numbers and so on.

$19, -8, \pi, 0.15, \frac{7}{5}, 2^3, \sqrt{16}, -3^2, (-3)^2, 0.99999..., 0.\dot{5}, \sqrt{5}$

Note: Numbers can belong to more than one number set.

Fact

Numbers can be classified as natural, integers, rational, irrational and real.

- **Natural numbers** are the numbers starting from 1 and increasing one by one to infinity. The set of natural numbers begins 0, 1, 2, 3, 4, …
- **Integers** are the set of positive and negative natural numbers. The set of integers is $\{..., -3, -2, -1, 0, 1, 2, 3, ...\}$
- **Rational numbers** are numbers that can be written as $\frac{a}{b}$, where a and b are integers and b is not zero. For example, $\frac{3}{2}$, 9.5, $3.\dot{2}$ are all rational numbers (i.e. fractions, decimals and repeating or terminating decimals).
- **Irrational numbers** are numbers that cannot be written as rational numbers. For example, $\pi, \sqrt{2}, \sqrt[3]{2}$
- **Real numbers** are all the numbers including natural numbers, integers, rational numbers and irrational numbers.

Fact

Natural numbers are also considered to be $\{1, 2, 3, ...\}$.

Fact

The dot above 2 in 3.2 means 3.222...

Worked example 1.1

Classify these numbers as natural, integer, rational, irrational and real. Remember that a number can belong to more than one classification.

$23, -11, \frac{\pi}{2}, 0.24, \frac{12}{15}, 3^3, \sqrt{36}, -5^2, (-10)^2, 0.999..., 0.\dot{4}, \sqrt{10}$

Solution

We can summarise the solution in a table.

Number	Natural	Integer	Rational	Irrational	Real
$23 = \frac{23}{1}$	Y	Y	Y	N	Y
$-11 = \frac{-11}{1}$	N	Y	Y	N	Y
$\frac{\pi}{2}$	N	N	N	Y	Y

Number	Natural	Integer	Rational	Irrational	Real
$0.24 = \frac{24}{100}$	N	N	Y	N	Y
$\frac{12}{15}$	N	N	Y	N	Y
$3^3 = 27 = \frac{27}{1}$	Y	Y	Y	N	Y
$\sqrt{36} = 6$	Y	Y	Y	N	Y
$-5^2 = -25$	N	Y	Y	N	Y
$(-10)^2 = 100$	Y	Y	Y	N	Y
$0.999\ldots = 1$	Y	Y	Y	N	Y
$0.\dot{4} = \frac{4}{9}$	N	N	Y	N	Y
$\sqrt{10}$	N	N	N	Y	Y

Fact

Natural numbers can be classified as prime or composite, odd or even, square numbers, cube numbers and so on.

- **Prime numbers** are numbers that have only two factors; one and themselves. For example, 2, 3, 5, 7, 11, 13, 17…
- **Composite numbers** are the numbers having more than two factors. For example, 4, 6, 8, 9, …
- **Even numbers** are the numbers divisible by 2. For example, 10, 52, 100, 7776.
- **Odd numbers** are the numbers having 1 as remainder when they are divided by 2. For example, 1, 3, 5, 21, 157.
- **Square numbers** are the numbers produced when a natural number is multiplied by itself. They are also called perfect squares. For example,| $1 = 1 \times 1$, $4 = 2 \times 2$, $81 = 9 \times 9$
- **Cube numbers** are the numbers produced when a natural number is multiplied by itself three times. For example, $8 = 2 \times 2 \times 2$

Worked example 1.2

Find the unknown digit in each case. List all the possible answers.

a The two-digit number $\boxed{8}\boxed{x}$ is a prime number.

b The two-digit number $\boxed{6}\boxed{a}$ is a square and a cube number.

Solution

Understand the problem

We need to find the unknown digit in each case.

Make a plan

a We know that a number is prime if it has no factors other than 1 and itself. Therefore, we can consider all prime numbers between 80 and 89.

b The two-digit number $\boxed{6}\boxed{a}$ is both a square and cube number. Consider all numbers between 60 and 69 that satisfy this condition.

Carry out the plan

a 83 and 89 are the only prime numbers between 80 and 89. So the unknown digit is $x = 3$ or $x = 9$

b If $a = 4$, the number 64 can be written as $64 = 8^2 = 8 \times 8$ and $64 = 4^3 = 4 \times 4 \times 4$

Look back

a All other digit values of x (not 3 or 9) make the two-digit number a composite number. For example, $80 = 2 \times 40$, $81 = 9 \times 9$, $84 = 4 \times 21$, $85 = 5 \times 17$, $86 = 2 \times 43$, $87 = 3 \times 29$, $88 = 8 \times 11$

b 64 is the only number between 60 and 69 that is both a perfect square (8^2) and perfect cube (4^3). There is no other value of a that satisfies these two conditions.

Reflect

Is there another way to approach this example? Can you say anything else about 64?

1.1.2 Properties of numbers

Explore 1.2

Can you find the digit a if the 3-digit number $\boxed{2}\boxed{3}\boxed{a}$ is divisible by 3 and 5?

Can you write the number 192 as the product of prime numbers?

Explore 1.3

Remember that decimals can be written as rational numbers or fractions.

For example, $0.75 = \frac{75}{100} = \frac{3}{4}$

Can you find the recurring decimal $3.\dot{4} = 3.444\ldots$ as a fraction?

Can you find the highest common factor (HCF) and the lowest common multiple (LCM) of 25 and 30.

Hint

Prime factorisation is writing a composite number as the product of prime numbers. For example, $12 = 2 \times 2 \times 3 = 2^2 \times 3$

Hint

The **highest common factor** (**HCF**), also called the **greatest common divisor** (**GCD**), is the largest common factor of two or more numbers. For example:

- the factors of 12 are **1**, 2, **3**, 4, 6, 12
- the factors of 15 are **1**, **3**, 5, 15

Therefore, the HCF of 12 and 15 is **3** (of the common factors 1 and 3).

Hint

The **lowest common multiple** (**LCM**) is the smallest common multiple of two or more numbers. For example:

- multiples of 12 are 12, 24, 36, 48, **60**, 72, …
- multiples of 15 are 15, 30, 45, **60**, 75, ….

Therefore, the LCM of 12 and 15 is **60**.

Hint

Divisibility rules: if number a is divisible by number b, and the remainder is zero, we say that '*a is divisible by b*'. Here are some divisibility rules.

Divisible by	Rule	Examples
2	The last digit of the number must be an even number	8, 86, 294
3	The sum of the digits must be divisible by 3	123 ($1 + 2 + 3 = 6$)
4	The number formed by the last two digits must be divisible by 4	112
5	The last digit must be 0 or 5	250, 3205
6	The number must be divisible by both 2 and 3 ($6 = 2 \times 3$)	234
8	The last three digits must be divisible by 8	1032
9	The sum of the digits must be divisible by 9	225 ($2 + 2 + 5 = 9$)
10	The last digit must be zero	280, 19 830

Worked example 1.3

Find possible values for digit x, if the three-digit number $\boxed{1}\boxed{2}\boxed{x}$ is divisible:

a by 3 **b** by 5 **c** by 4 **d** by 3 and 5.

Challenge Qd

Solution

We need to find x.

Apply divisibility rules of 3, 5 and 4.

a $\boxed{1}\boxed{2}\boxed{x}$ is divisible by 3 only if the sum of the digits is divisible by 3. That is, $1 + 2 + x = 3 + x$ must be divisible by 3. Therefore, possible x-values are 0, 3, 6 or 9.

b $\boxed{1}\boxed{2}\boxed{x}$ is divisible by 5 if x is 0 or 5.

c $\boxed{1}\boxed{2}\boxed{x}$ is divisible by 4 if the two-digit number formed by the last 2 digits, $\boxed{2}\boxed{x}$, is divisible by 4. So, possible x-values are 0, 4 and 8.

d If $\boxed{1}\boxed{2}\boxed{x}$ is divisible by both 3 and 5, then we can use the common solutions of parts a and b. Therefore, the only possible value of x is 0.

Looking back, we can see that all answers do make sense.

a $120 = 1 + 2 + 0 = 3$, $123 = 1 + 2 + 3 = 6$, $126 = 1 + 2 + 6 = 9$, $129 = 1 + 2 + 9 = 12$ are all divisible by 3.

b $120 = 5 \times 24$ and $125 = 5 \times 25$ are divisible by 5.

c $120 = 4 \times 30$, $124 = 4 \times 31$, $128 = 4 \times 32$ are all divisible by 4.

d $120 = 3 \times 40 = 5 \times 24$ and divisible by 3 and 5.

Reflect

Note that 3 and 5 are both prime numbers. If a number is divisible by 3 and 5, can we conclude that the number is also divisible by 15?

Worked example 1.4

Find the values for the digits x and y:

a $120 = 2^x \times 3 \times y$ b $54 = 3^x \times 4^y$

Challenge Qb

Solution

We need to find x and/or y.

Both numbers may be written as the product of prime numbers.

a $120 = 2 \times 60 = 2 \times 2 \times 30 = \ldots = 2^3 \times 3 \times 5 = 2^x \times 3 \times y$
Therefore, $x = 3$, $y = 5$

b $54 = 2 \times 27 = \ldots = 3^x \times 4^y$, thus $3^3 = 3^x$ where $x = 3$ and

$2^1 = 4^y = (2^2)^y = 2^{2y}$ so $1 = 2y$ and $y = \dfrac{1}{2}$

Looking back:

a If we substitute $x = 3$ and $y = 5$ into $2^x \times 3 \times y$, the result will be 120.

b If we substitute $x = 3$ and $y = \dfrac{1}{2}$ into $3^x \times 4^y$, since $4^{\frac{1}{2}} = \sqrt[2]{4^1} = 2$, then $3^3 \times 4^{\frac{1}{2}} = 27 \times 2 = 54$

Worked example 1.5

Write the recurring decimal 3.4444… as a fraction.

Solution

First change the recurring decimal to a simple decimal by removing the recurring part.
As only one digit (the 4) is repeated, we multiply by 10.

Hint

A **recurring decimal** is a decimal where one decimal digit or set of decimal digits repeats forever. Recurring decimals can be written as fractions.

Hint

This method may appear to be a 'trick', but you will learn other methods in later years.

Let $x = 3.444\ldots$

Multiply both sides by 10 to give $10x = 34.444\ldots$

so,

$10x = 34.444\ldots,$

$x = 3.444\ldots$, and if we subtract both equations, we have

$9x = 31$

or $x = \dfrac{31}{9}$

Looking back, if we divide 31 by 9, it gives 3.444…, which shows that the solution makes sense.

Hint

We can **convert fractions** into percentages and decimals. For example, $\frac{13}{20}$ can be written as both 65% and 0.65.

The table below shows some other conversion examples.

Fraction	Decimal	Percentage
$\frac{6.5}{10}$	0.65	65%
$\frac{11}{25}$	0.44	44%
$\frac{60}{75}$	0.8	80%
$\frac{23}{50}$	0.46	46%
$\frac{21}{28}$	0.75	75%
$\frac{22}{20}$	1.10	110%
$\frac{17}{18}$	0.9444…	94.444…%

1.1.3 Operations with numbers

Explore 1.4

Write each expression as a single number.

a $24 \div 8 \times 3$ **b** $2.\dot{3} - 1.\dot{2}$ **c** $\frac{2}{3} + \frac{3}{2} - 1\frac{1}{6}$

d $2\frac{3}{4} \times 1\frac{1}{2}$ **e** $4 - 3^2 \div (-3) \times 3 - (2 - 15)$

Hint

The **order of operations** rule helps you to work out an expression correctly when there is more than one operation involved. Here is the rule to solve mixed operations:

1. (**P**) Do the parenthesis first
2. (**E**) Exponents
3. (**MD or DM**) Do the multiplication or division from left to right
4. (**AS or SA**) Addition or subtraction from left to right

You can remember this rule as **PEMDAS** or **PEDMAS**.

Worked example 1.6

Find the solutions to these calculations.

a $32 \div 8 \times 4$ **b** $4.\dot{2} - 3.\dot{5}$ **c** $\frac{3}{5} + \frac{5}{3} - 2\frac{3}{15}$

d $1\frac{2}{3} \times 2\frac{3}{5}$ **e** $8 - 2^4 \div (-4) \times 2 - (3 - 5)$

Solution

a $32 \div 8 \times 4 = 4 \times 4 = 16$

From left to right first divide then multiply.

b $4.\dot{2} - 3.\dot{5} = \frac{38}{9} - \frac{32}{9} = \frac{6}{9} = 0.\dot{6}$

First simplify the recurring decimals, then subtract.

c $\frac{3}{5} + \frac{5}{3} - 2\frac{3}{15} = \frac{9}{15} + \frac{25}{15} - \frac{33}{15} = \frac{1}{15}$

First find the lowest common denominator and, simplify the mixed numbers, then add and subtract.

d $1\frac{2}{3} \times 2\frac{3}{5} = \frac{5}{3} \times \frac{13}{5} = \frac{65}{15} = \frac{13}{3}$

Change the mixed numbers into fractions then multiply.

e $8 - 2^4 \div (-4) \times 2 - (3 - 5)$

$= 8 - 16 \div (-4) \times 2 - (-2)$ Exponents and parenthesis first.

$= 8 - (-4) \times 2 + 2$ Division first from the left.

$= 8 - (-8) + 2 = 8 + 8 + 2 = 18$ Multiplication then division.

Reflect

In a class discussion, some students might give two different answers for $2^4 \div 2^3 \times 2$. Can you explain how that might happen?

Practice questions 1.1

1 Classify the numbers in the table.

Number	Natural	Integer	Rational	Irrational	Real
101					
-6					
π^2					
2.13					
$\frac{19}{3}$					
-2^3					
$\sqrt{6}$					
-5^2					
$(-1.1)^2$					
0.111…					
$3.\dot{5}$					
$\sqrt{64}$					

Challenge Q2c

Challenge Q2d

2 a How many prime numbers are there between 25 and 65?

b How many square numbers are there between 1 and 50?

c How many numbers between 1 and 100 are both square and cube numbers?

d How many numbers between 1 and 100 include 9 as a digit?

3 a What is the HCF and the LCM of the numbers 12 and 30?

b Write 540 as product of prime numbers.

c Find possible values of variable x, if the three-digit number $\boxed{4}\boxed{x}\boxed{6}$ is divisible by 6.

Challenge Q2d

d Is 2341 a prime number? Explain your answer.

4 Simplify each expression.

a $\dfrac{\frac{1}{2}+\frac{2}{3}}{\frac{1}{2}-\frac{2}{3}}$ b $\frac{3}{4}\times 1\frac{4}{5}$ c $1\frac{12}{13}\div\frac{5}{26}$ d $\left(\frac{3}{5}\right)^2\div\frac{1}{2}\times\frac{5}{2}$

5 Convert these recurring decimals to fractions.

a 2.333… b 3.222… c 11.2333… d 5.121212…

6 Perform the following operations.

a $121.2 + 1.212 - 12.12$ b $2 \times 30 \times 12$

c $(2 \times 3)^2$ d $\sqrt{1.44}$

7 Simplify the following expressions.

a $\{2^3 \div (-2)\}\{(-2) - 3^2\}$ b $(16 - 3^3 \times 2 \div (-6)) - 25$

c $\sqrt{81} \div 3^2 - 3$ d $\dfrac{-\frac{5}{4}+1\frac{2}{3}}{2}\div\dfrac{11}{24}$

1.2 Algebra

1.2.1 Algebra basics

Explore 1.5

Explore the following questions.

a If $x = 3$, can you find the value of:

i $2x$ **ii** x^3 **iii** $2x + 5x$ **iv** $11x - 7x$

b Can you find the missing x- and y-values for the rule $x + 2y = 10$?

x	2		6	
y		1		5

c Can you find the next three numbers in each number sequence?

i 2, 4, 6, 8, … **iii** 3, 5, 7, 9, …

iii 1, 4, 9, 25, … **iv** 1, 1, 2, 3, 5, 8, …

Challenge Qciv

When you don't know a number in a context, you can use a letter to represent the number; for example, *x, y, z, a, b, c,* etc. These letters are called **variables**.

Combinations of variables with different operations are called (**algebraic**) **expressions**; for example, $2x + 5y$ or $2xy - 6x^2$

Each part of the expression separated by a + sign or a − sign is called a **term**; for example, $2x$ and $5y$ are terms in the expression $2x + 5y$

If there is a rule or relationship between one or more sets of numbers, it is called a **number pattern**. For example, $y = 2x$ is a number pattern rule where the y-values are double the x-values.

If there is a rule within a set of numbers then that set is called a **number sequence**. For example, {3, 6, 9, 12, …} is the sequence of positive multiples of 3.

Worked example 1.7

Find the value of each expression when $z = 5$

a $5z$ **b** z^3 **c** $2z + z^2$ **d** $4z \div 3z$

Solution

Substitute $z = 5$ into each expression.

a $5(5) = 25$ **b** $(5)^3 = 5 \times 5 \times 5 = 125$

c $2(5) + 5^2 = 10 + 25 = 35$ **d** $4(5) \div 3(5) = 20 \div 15 = \frac{4}{3}$ or 1.333…

Worked example 1.8

Find the missing variable values, if $y = 3x - 2$

x	1		−2	
y		4		7

Solution

We need to find the missing x- or y-values in the table which satisfy the equation $y = 3x - 2$

Substitute the given x- or y-values into the rule $y = 3x - 2$ to find the missing variable.

When $x = 1$, $y = 3(1) - 2$
$y = 1$

When $y = 4$, $4 = 3x - 2$
$6 = 3x$
$2 = x$

When $x = -2$, $y = 3(-2) - 2$
$y = -6 - 2$
$y = -8$

When $y = 7$, $7 = 3x - 2$
$9 = 3x$
$3 = x$

Looking back, when the x- and y-values have been found and are substituted in $y = 3x - 2$, the equation will be correct.

When $x = 1$, $y = 1$, then $1 = 3 \times (1) - 2 = 3 - 2 = 1$ (true)

When $x = 2$, $y = 4$, then $4 = 3 \times (2) - 2 = 6 - 2 = 4$ (true)

When $x = -2$, $y = -8$, then $-8 = 3(-2) - 2 = -6 - 2 = -8$ (true)

When $x = 3$, $y = 7$, then $7 = 3(3) - 2 = 9 - 2 = 7$ (true)

Investigation 1.1

Exploring the number pattern

1 Use the table to answer the questions below.

x	y
1	3
2	6
3	9
4	12
…	…
…	…
…	…

a Find the missing x- and y-values.

b Find a rule describing the relationship between x and y.

c Show that your rule works for all x- and y-values.

2 Look at the table below and answer the questions.

a	b
1	5
2	8
3	11
4	14
…	…
…	…
…	…

a Find the missing a and b values in the table.

b Describe a rule that connects a and b and justify that rule.

Fact

If the difference between consecutive terms in a number sequence is constant, then the number sequence has a **linear pattern**.
For example, the sequence 5, 8, 11, … has a constant difference:
$8 - 5 = 11 - 8 = 3$, so there is a linear pattern.

Connections

Linear patterns can be described by straight lines in a coordinate plane. You will investigate linear patterns and their graphs in Chapter 5: Coordinate geometry.

1.2.2 Operations with algebraic expressions

Explore 1.6

Can you work out each of the following? Give reasons for your answers.

1 − $6a + 3a$ 2 − $7x - 2x$ 3 − $\dfrac{14a}{2a}$ 4 − $3x \times 6x \div 2x$

Here are some reminders of rules for operating on algebraic expressions:

1 You can only add or subtract **like terms**; that is, terms that have the same variable. For example, let's add the two expressions $(2x + 3y - 5)$ and $(5x - 3y + 2)$.

$2x + 3y - 5 + 5x - 3y + 2$

Add the x terms and the y terms separately:

$(2x + 5x) + (3y - 3y) - 5 + 2$

This simplifies to $7x - 3$

2 When you multiply or divide algebraic expressions, you multiply the **coefficients** first, then the variables. You can multiply or divide terms that do not have the same variables.

Let's multiply $12x$ by $6x$, then divide $12x$ by $6x$.

$12x \times 6x = 72x^2$ and $\dfrac{12x}{6x} = \dfrac{12}{6} \times \dfrac{x}{x} = 2 \times 1 = 2$

Worked example 1.9

Simplify each of these expressions.

a $8t - 12w + 5t + 6w$

b $-2ab + 5ab - (-ab)$

c $3a \times 4b \times 2c$

d $-3a \times 2ab \times 4b$

e $10m \div 5m$

f $12ab \div (-2a)$

Solution

Apply the rules of operations.

a $8t - 12w + 5t + 6w = (8t + 5t) + (-12w + 6w)$
$= 13t + (-6w) = 13t - 6w$

b $-2ab + 5ab - (-ab) = -2ab + 5ab + ab$
$= (-2 + 5 + 1)ab = 4ab$

c $3a \times 4b \times 2c = (3 \times 4 \times 2)(a \times b \times c) = 24abc$

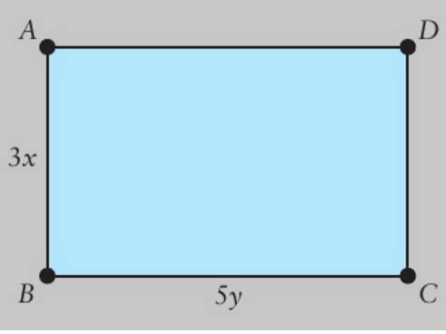
d $-3a \times 2ab \times 4b = (-3 \times 2 \times 4)(a \times ab \times b) = -24a^2b^2$

e $10m \div 5m = (10 \div 5)(m \div m) = 2$

f $12ab \div (-2a) = (12 \div (-2))(ab \div a) = 6b$

Worked example 1.10

Find the perimeter and area of the rectangle $ABCD$.



Solution

We need to find the perimeter and area of the rectangle $ABCD$. Remember opposite sides have equal measures.

Use the area and perimeter formulae for a rectangle.

$P = 2(a + b)$, $A = a \times b$ where a and b are the side measurements.

$P = 2(3x + 5y) = 6x + 10y$ $A = 3x \times 5y = 15xy$

Looking back, the perimeter is the sum of the lengths of the sides. Therefore, $3x + 3x + 5y + 5y = 6x + 10y$ which is what we got from the formula.

The area of a rectangle is the product of the length and width. Thus, $3x \times 5y$ gives $15xy$.

1.2.3 Equations

Explore 1.7

Can you answer the following questions? Support your response with an example.

What is an equation?

Is there any difference between an algebraic expression and equation?

What is your strategy to solve the equation $2x + 5 = 19$?

An **equation** is an algebraic expression that includes an '=' sign.

$3x + 5$ is an algebraic expression, while $3x + 5 = 18$ is an equation.

An **inverse operation** is an operation that gives the original number when it is applied to the answer of the operation. Here are some of the operations and their inverse operations.

- Subtraction is the inverse of addition.
- Division is the inverse of multiplication.
- A square root is the inverse of a square.

In order to solve an equation, we can apply the inverse operation to both sides until the value of the variable is found.

The order of the operations is important. Always use inverse operations in the reverse of the order of the operations applied to the variable.

Worked example 1.11

Solve the following equations.

a $x + 7 = 19$ **b** $y - 9 = 27$ **c** $5 \times z = 45$ **d** $\frac{w}{5} = 11$

Solution

The table shows the steps for the solutions.

Equation	Operation	Inverse operation	Solution
$x + 7 = 19$	Added 7	Subtract 7	$x + 7 - 7 = 19 - 7$ So $x = 12$
$y - 9 = 27$	Subtracted 9	Add 9	$y - 9 + 9 = 27 + 9$ So $y = 36$
$5 \times z = 45$	Multiplied by 5	Divide by 5	$\frac{5 \times z}{5} = \frac{45}{5}$ So $z = 9$
$\frac{w}{5} = 11$	Divided by 5	Multiply by 5	$\frac{w}{5} \times 5 = 11 \times 5$ So $w = 55$
$k^2 = 49$	Squared	Square root	$\sqrt{k^2} = \sqrt{49}$ So $k = 7$

Worked example 1.12

Solve these two-step equations.

a $12 + 3x = 24$

b $2y - 7 = 21$

c $\frac{z + 9}{4} = 7$

d $\sqrt{w} - 11 = 25$

Solution

We need to solve each equation to find the variables x, y, z or w.

Identify the two operations in order. When we know the order of the operations, we can perform the inverse of each operation, starting from the last one. For example, in part a, x is multiplied by 3 and then 12 is added. Thus, the inverse operations are subtract 12, then divide by 3.

a $12 + 3x - 12 = 24 - 12$

$3x = 12$

$\frac{3x}{3} = \frac{12}{3}$

$x = 4$

b $2y - 7 + 7 = 21 + 7$

$2y = 28$

$\frac{2y}{2} = \frac{28}{2}$

$y = 14$

c $\frac{z+9}{4} \times 4 = 7 \times 4$

$z + 9 = 28$

$z + 9 - 9 = 28 - 9$

$z = 19$

d $\sqrt{w} - 11 + 11 = 25 + 11$

$\sqrt{w} = 36$

$(\sqrt{w})^2 = 36^2$

$w = 1296$

Substitute each solution into the original equations to see if the solution is valid.

a $12 + 3(4) = 12 + 12 = 24$ (true)

b $2(14) - 7 = 28 - 7 = 21$ (true)

c $\frac{(19) + 9}{4} = \frac{28}{4} = 7$(true)

d $\sqrt{1296} - 11 = 36 - 11 = 25$(true)

Reflect

Can you think of another method to solve these equations?

Have a look at the following approach. It is called the **deductive** approach.

$12 + 3x = 24$

$12 + 12 = 24$, therefore $3x = 12$

$3 \times 4 = 12$, thus $x = 4$

Try to apply this approach to the other questions you solved.

Which method do you prefer? Why?

Activity

Why did the tooth get dressed up?

Find the answer for each part. Then put the letter in the correct place in the table on the next page to find why the tooth got dressed up.

For example, with T: $x + 8 = 15$, we solve to find that $x = 7$.

Then, find 7 in the table and put a T above it.

T: $x + 8 = 15$	A: $a = -4 - 6$	A: $a = \left(\frac{1}{2}\right)^3$	T: $3 + x = 1$
T: $8m = -56$	D: $11x = 121$	A: $x + x = 1$	T: $75 \div a = 15$
E: $0.2x = -1$	E: $8a = 0$	S: $8 + x = 12$	H: $1 - x = 4$
U: $60 - n = 0$	O: $35 \div n = 1$	N: $5(x - 9) = 0$	T: $n = 20 - n$
U: $7 - x = 8$	O: $\frac{100}{m} = 5$	O: $6 \div x = 2$	W: $\frac{1}{4} + n = 1$
T: $7a = 7$	I: $-3 - 3 = a$	K: $1 - x = -1$	S: $4x = 1$
B: $-a = -8$	T: $a - 8 = 2a$	I: $\frac{x}{3} = 2$	T: $10 - n = -2$
E: $\frac{m}{20} = 5$			

						T								
5	−3	0	11	100	9	7	−6	4	−2	$\frac{3}{4}$	$\frac{1}{8}$	$\frac{1}{4}$	−10	8

3	60	1	−8	20	−7	$\frac{1}{2}$	2	−5	6	10	35	−1	12

Fact

If the equations are more complex, then the following rules can be applied:

- If there are variables on both sides of an equation, you should collect all the variables on one side and follow the inverse operation rule.
- If there are parentheses in an equation, always start by expanding the parentheses.

Worked example 1.13

Solve these equations.

a $2m + 11 = 7m - 4$ **b** $5(t - 3) = 4(t + 2) - 7$

Solution

We need to solve the first equation for m, and the second equation for t.

For part a we will collect the terms with the variables on one side and the terms that are constants on the other side. For part b we will expand the parentheses and collect the variables on one side and the numbers on the other.

a $2m + 11 = 7m - 4$

$2m + 11 - 2m = 7m - 4 - 2m$

$11 + 4 = 5m - 4 + 4$

$15 = 5m$

$3 = m$

b $5(t - 3) = 4(t + 2) - 7$

$5t - 15 = 4t + 8 - 7$

$5t - 15 - 4t + 15 = 4t + 8 - 7 - 4t + 15$

$t = 16$

Look back: substitute the answers into the original equations to check.

a $2(3) + 11 = 7(3) - 4$

$6 + 11 = 21 - 4$

$17 = 17$ (true)

b $5((16) - 3) = 4((16) + 2) - 7$

$5 \times 13 = 4 \times 18 - 7$

$65 = 72 - 7$

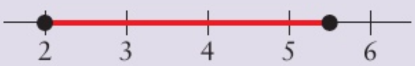
$65 = 65$ (true)

1.2.4 Inequalities

Explore 1.8

Recall the inequality signs: ($<$, $>$, $\leq$ and $\geq$). Can you define each sign?

How can we represent the following lines as inequalities?



How do we solve inequalities? Do we need different methods for solving inequalities than for solving equations?

An **inequality** is where the equation sign in an algebraic statement is replaced with one of the **inequality signs**:

$<$ less than

$>$ greater than

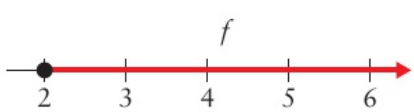
$\leq$ less than or equal to

$\geq$ greater than or equal to

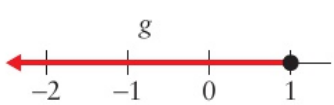
For example, $3x + 5 > 4x - 3$ is an inequality.

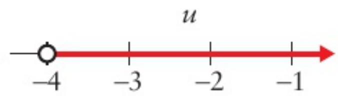
Solutions of equations are single points on a number line, but solutions of inequalities are sets of numbers on a number line. For example:

$f \geq 2$ can be represented on a number line as

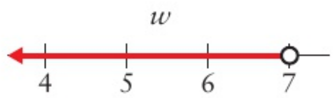


$g \leq 1$ can be represented on a number line as



$u > -4$ can be represented as  .

Note that -4 is represented with an open circle, since $u = -4$ is not part of the solution.

$w < 7$ can be represented as  where 7 is not part of the solution.

Worked example 1.14

Solve each inequality and graph the solution on a number line.

a $x + 7 > 12$ **b** $y - 9 < 2$

c $5z \geq 45$ **d** $\frac{w}{5} \leq 3$

Solution

a $x + 7 > 12$

$x + 7 - 7 > 12 - 7$

$x > 5$

All numbers greater than 5. The circle at 5 is an open circle, which indicates that 5 is not included.

b $y - 9 < 2$

$y - 9 + 9 < 2 + 9$

$y < 11$

All numbers less than 11. The 11 is not included.

c $5z \geq 45$

$\frac{5z}{5} \geq \frac{45}{5}$

$z \geq 9$

All numbers greater than or equal to 9. The 9 is included.

d $\frac{w}{5} \leq 3$

$\frac{w}{5} \times 5 \leq 3 \times 5$

$w \leq 15$

All numbers less than or equal to 15. The 15 is included.

Fact

- We can use the same rules to solve an inequality as we do to solve an equation.
- There is only one exception: when multiplying or dividing an inequality by a negative number, the inequality sign must be reversed.

For example: $18 > -6$

If we divide both sides by -3 we get $-6 > 2$, which is incorrect.

Reversing the sign when dividing by a negative number gives the correct answer of $-6 < 2$

Worked example 1.15

Solve the inequality and graph the solution on a number line.

$2(x + 3) - 5(2 + x) \leq 23$

Solution

We need to solve the inequality for x and graph it on a number line.

Expand the parentheses, collect the variables and use the inverse operation method.

$2(x + 3) - 5(2 + x) \leq 23$

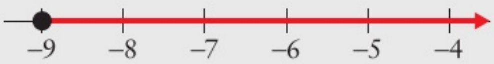
$2x + 6 - 10 - 5x \leq 23$

$-3x - 4 \leq 23$

$-3x \leq 27$

$\frac{-3x}{-3} \leq \frac{27}{-3}$

$x \geq -9$



Looking back, our solution is valid since any point greater than or equal to -9 satisfies the inequality. Let's choose -9 and -5 to check the solution.

When $x = -9$: $2((-9) + 3) - 5(2 + (-9)) \leq 23$

$2(-6) - 5(-7) \leq 23$

$-12 + 35 \leq 23$

$23 \leq 23$ (true)

When $x = -5$: $2((-5) + 3) - 5(2 + (-5)) \leq 23$

$2(-2) - 5(-3) \leq 23$

$-4 + 15 \leq 23$

$11 \leq 23$ (true)

Practice questions 1.2

1 Complete the table for each of these rules.

a $y = 3x + 2$

x	−2	0	2	4
y				

b $w = 5z - 4$

z	−1	1	3	5
w				

2 Find the rule for each of these patterns.

a

x	0	1	2	3
y	1	3	5	7

b

x	1	2	3	4
y	5	8	11	14

3 Simplify each algebraic expression.

a $2x + 3x + 5x$ b $3y - 4y + 7y$ c $2a - 5b + 3a + 7b$

d $-3x \times 7$ e $\dfrac{36w}{9}$ f $45ab \div 5b$

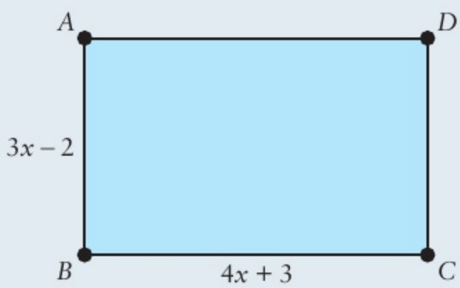
4 Simplify each algebraic fraction.

a $\dfrac{3x}{2} - \dfrac{x}{3}$ b $\dfrac{x}{3} + \dfrac{x}{4}$ c $\dfrac{2x}{3} \times \dfrac{4x}{5}$ d $\dfrac{6x}{5} \div \dfrac{2x}{3}$

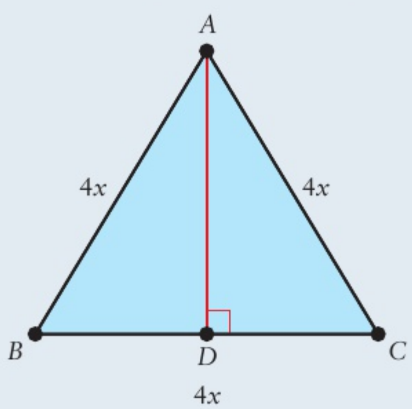
Challenge Q5

5 Find the perimeter and the area of these shapes as an algebraic expression.

a *ABCD* is a rectangle



b *ABC* is an equilateral triangle.



6 Solve each of these equations.

a $2x + 5 = 27$

b $\frac{x-3}{5} = 4$

c $2(x - 1) + 3(x + 1) = 21$

d $2(3x - 1) = 5(x + 2)$

7 Solve each inequality and graph the solution on a number line.

a $2x - 1 \leq 7$

b $\frac{x-3}{4} \geq 1$

c $5x - 7 > 3x + 11$

d $\frac{x}{2} - \frac{x}{3} < 1$

8 Write an equation for each of these problems and then solve it.

a The product of a number x and 12 is 96. Find x.

b When 3 is added to the product of 5 and a number y, the answer is 38. Find the number y.

c The sum of three consecutive numbers is 96. Find the middle number.

d Eric is 3 years older than his sister Mette. If the sum of their ages is 27, find their ages.

1.3 Geometry

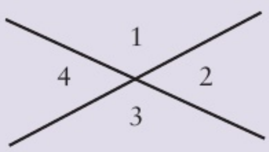
1.3.1 Geometry basics

Explore 1.9

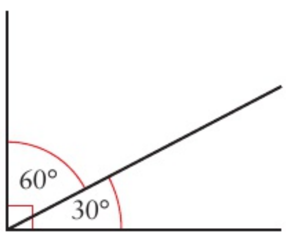
Two adjacent angles make a right angle. If one measures 34°, what is the value of the other? Can you draw a diagram to illustrate this?

What is a straight angle?

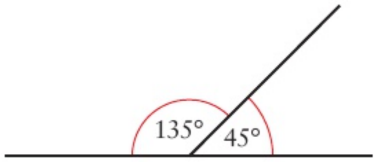
List all possible connections between the values of the angles in the diagram.



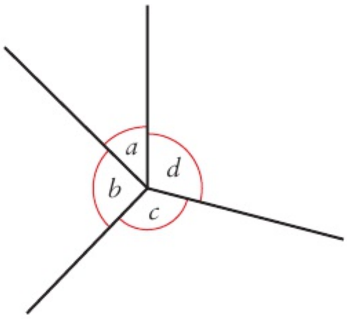
- If two adjacent angles add up to 90°, they make a **right angle**. The angles are known as **complementary angles**. These angles are complementary and form a right angle since 30° + 60° = 90°



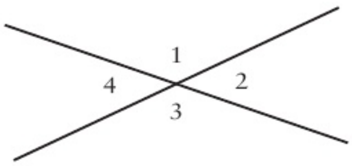
- If the sum of two adjacent angles is 180°, they make a **straight angle**. The angles are known as **supplementary angles.** These angles are supplementary and form a straight angle since 135° + 45° = 180°



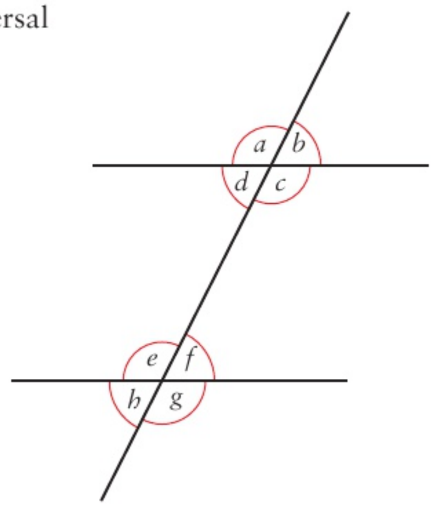
- If a set of adjacent angles makes a complete revolution, the angles must add up to 360°. The sum of the angles at a point is 360°: $a + b + c + d = 360°$



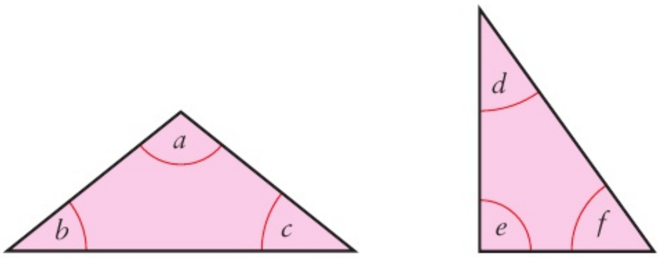
- If two lines intersect at a point, they form four adjacent angles. An angle and its neighbour are supplementary angles. Angles 1 and 3 or 2 and 4 have equal value and are called **vertically opposite** angles.



- If two parallel lines are cut by a transversal (a line intersecting both parallel lines) then eight angles are formed.
  Angles a and e, b and f, c and g, d and h are **corresponding angles**. They have equal measures.
  Angles d and f, c and e are **alternate (interior) angles**. They have equal measures.
  Angles a and g, b and h are **alternate (exterior) angles**. They have equal measures.
  Angles d and e, c and f are **co-interior angles** and they are supplementary angles.



- The sum of interior angles of a triangle is 180°.



$a + b + c = 180°$ $\qquad$ $d + e + f = 180°$

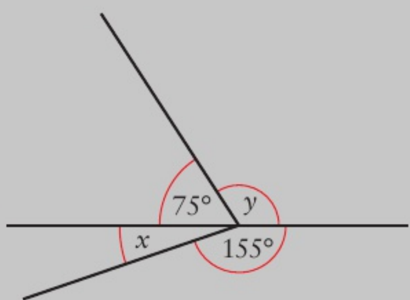
- The sum of interior angles of a quadrilateral is 360°.

$g + h + j + i = 360°$ $\qquad$ $k + l + n + m = 360°$

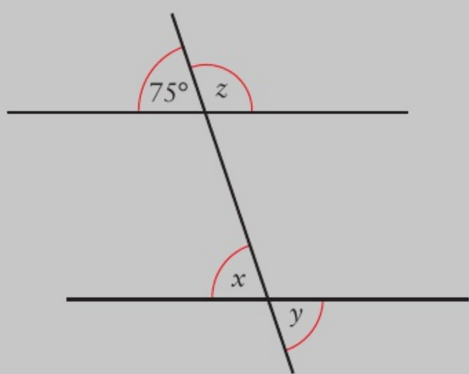
Worked example 1.16

Find the missing angles in each diagram. Explain your solution.

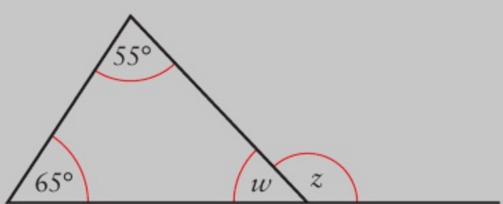
a



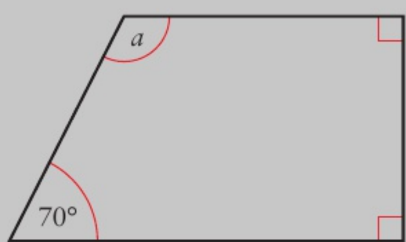
b



c



d



Solution

Understand the problem

We need to find the missing angles in each diagram.

Make a plan

a Use the supplementary angle property, and/or the sum of angles at a point.

b Use corresponding, supplementary and vertically opposite angle properties.

c The sum of the interior angles in a triangle is 180°.

d The sum of the interior angles in a quadrilateral is 360°.

Carry out the plan

a $75° + y = 180°$ (supplementary angles), so $y = 180° - 75° = 105°$
$155° + x = 180°$ (supplementary angles), so $x = 180° - 155° = 25°$

b $75° + z = 180°$ (supplementary angles), so $z = 105°$

$x = 75°$ (corresponding angles)

$y = x = 75° = y$ (vertically opposite angles)

c $55° + 65° + w = 180°$ (sum of interior angles of a triangle)
$120° + w = 180°$, so $w = 60°$
$60° + z = 180°$ (supplementary angles), so $z = 120°$

d $a + 90° + 90° + 70° = 360°$ (sum of interior angles in a quadrilateral is 360°)
$a + 250° = 360°$, so $a = 110°$

Look back

All the solutions are correct.

a If we substitute values of x and y, the four adjacent angles around the central point form a complete revolution:
$75° + 105° + 155° + 25° = 360°$

b The two horizontal lines are parallel, and the transversal cuts both lines, so they form angles $x = 75°$ (corresponding angle) and
$x = y = 75°$ (vertically opposite angles)

$z = 180° - 75° = 105°$ (supplementary angles)

c If we substitute $w = 60°$ in the sum $55° + 65° + w$, we get 180° (correct)

If we substitute $z = 120°$ in the sum $60° + z = 180°$ (supplementary angles), then we get 180° (correct)

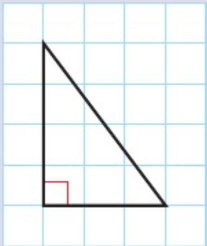
d Substitute $a = 110°$ into the sum of interior angles:
$110° + 90° + 90° + 70° = 360°$ (true)

1.3.2 Perimeter and area

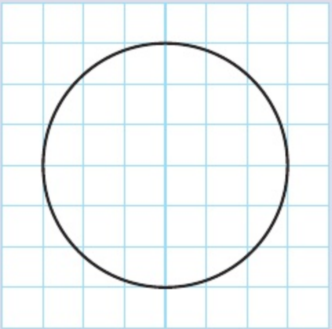
Explore 1.10

a Can you find the perimeter of each shape?

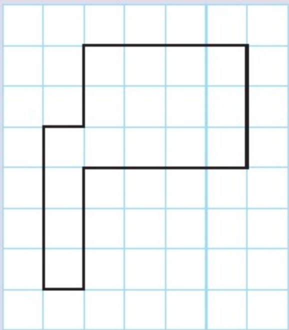
i



ii

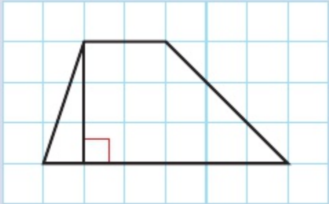


iii

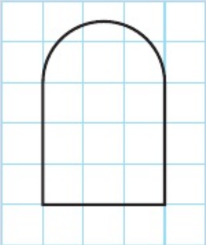


b Can you find the area of each shape?

i

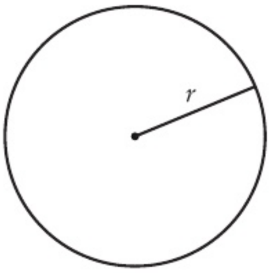


ii



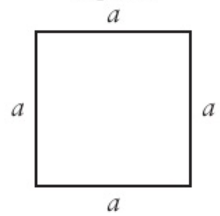
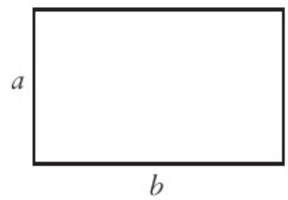
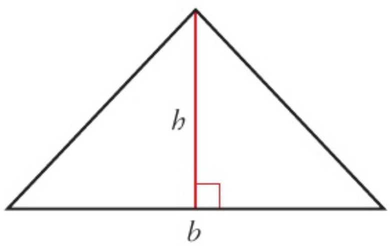
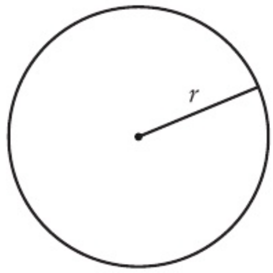
The **perimeter** P of an object is the sum of the lengths of all its sides.

The perimeter of a circle, called the **circumference** C, is given by $C = 2 \times \pi \times r$, where r is the radius of the circle.



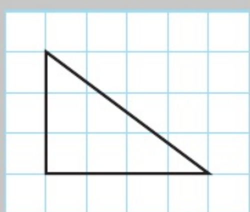
The **area** A of an object is the space it covers in 2-dimensional space.

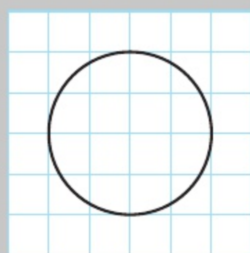
The area is calculated by the relevant formula. This table summarises some of the formulae.

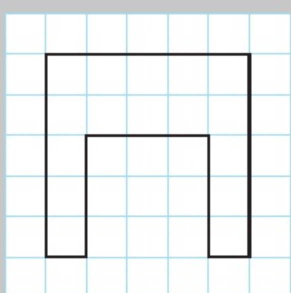
Square  $A = a \times a = a^2$	Rectangle  $A = a \times b$
Triangle  $A = \frac{1}{2}bh$	Circle  $A = \pi \times r^2$

Worked example 1.17

Find the perimeter of the following shapes.

a 

b 

c 

Solution

Understand the problem

We need to find the perimeter of a triangle and a composite shape, and the circumference of a circle.

Make a plan

a In order to find the perimeter of the triangle, we need to find the length of the hypotenuse. We will use Pythagoras' theorem.

b Apply the circumference formula where $r = 2\,\text{cm}$.

c Find the sum of the lengths around the composite shape.

Carry out the plan

a Let c = the hypotenuse. $c^2 = 3^2 + 4^2 = 9 + 16 = 25$, $c = 5$
Thus, $P_{\text{triangle}} = 3 + 4 + 5 = 12$ units.

b $C = 2 \times \pi \times r$, $C = 2 \times \pi \times 2 = 4\pi$ units.

c $P = 5 + 1 + 3 + 3 + 3 + 1 + 5 + 5 = 26$ units

Look back

a Only the hypotenuse is missing for the perimeter. Pythagoras' theorem helps us to find that the hypotenuse measures 5 units. So, $P = 12$ units

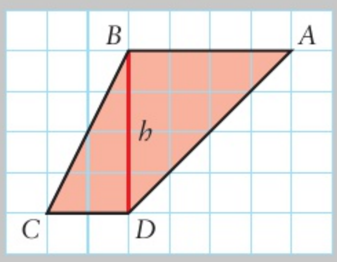
b The circumference of the circle is $2 \times \pi \times 2 = 4\pi$ units

c The perimeter of the composite shape is the sum of the lengths around the shape, which is 26 units.

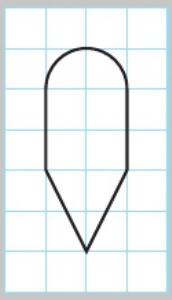
Worked example 1.18

Find the area of the following shapes. Explain your solution.

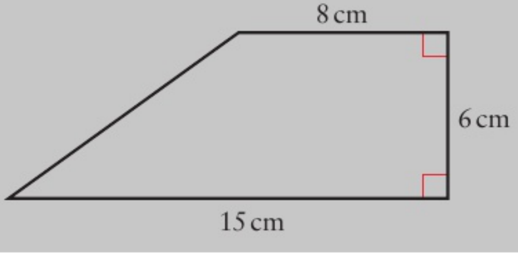
a



b



c



Solution

Understand the problem

We need to find the area of each shape.

Make a plan

a This shape is a combination of two triangles. The area is the sum of the areas of the two triangles.

b The composite shape can be divided into a semicircle, a triangle and a square.

c The trapezium can be divided into a rectangle and a triangle.

Carry out the plan

a A = area of BDC + area of DBA

$$= \frac{1}{2} \times 2 \times 4 + \frac{1}{2} \times 4 \times 4$$

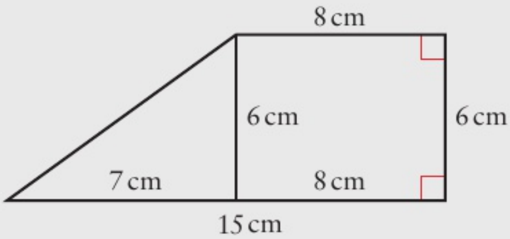
$$= (4 + 8) = 12 \text{ units}^2$$

b A = area of semicircle + area of square + area of triangle

$$= \frac{1}{2} \times \pi \times 1^2 + 2 \times 2 + \frac{1}{2} \times 2 \times 2$$

$$= \left(\frac{\pi}{2} + 6\right) \text{ units}^2$$

c



$$A = \frac{1}{2} \times 7 \times 6 + 6 \times 8$$

$$= 21 + 48 = 69 \text{ units}^2$$

Each of the composite shapes is divided into known shapes to find the total area.

Look back

a The quadrilateral is divided into two triangles and the total area is $4 + 8 = 12$ units^2

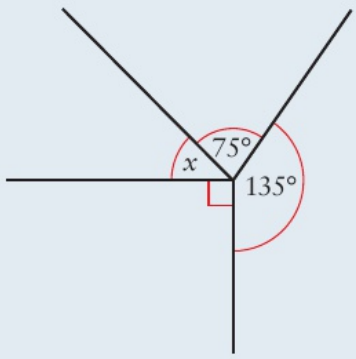
b The composite shape is divided into a semicircle, a square and a triangle, and the total area is $\frac{\pi}{2} + 4 + 2 = \left(\frac{\pi}{2} + 6\right)$ units^2

c The trapezium is divided into a triangle and a rectangle and the total area is $21 + 48 = 69$ units^2

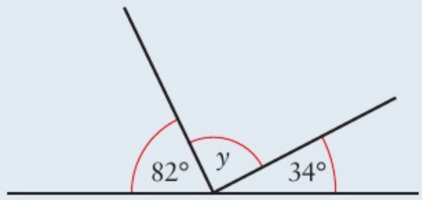
Practice questions 1.3

1 Find the value of the variable in each case.

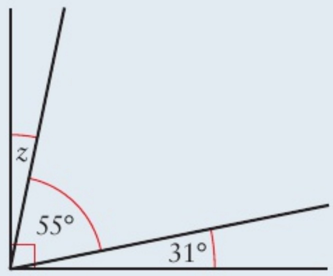
a



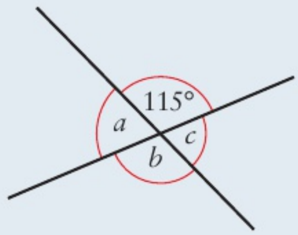
b



c

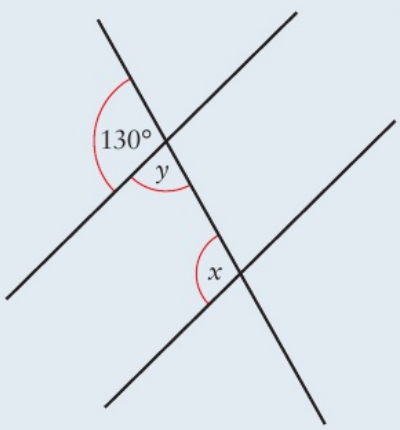


d

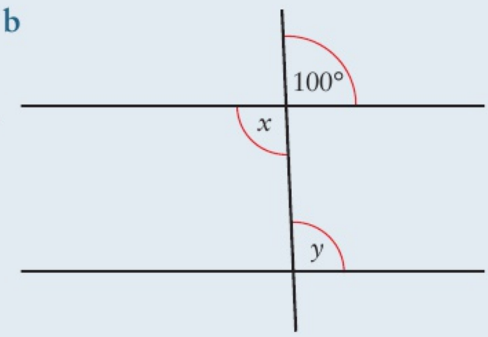


2 Each diagram contains two parallel lines and a transversal. Find the value(s) of the variable(s) in each case.

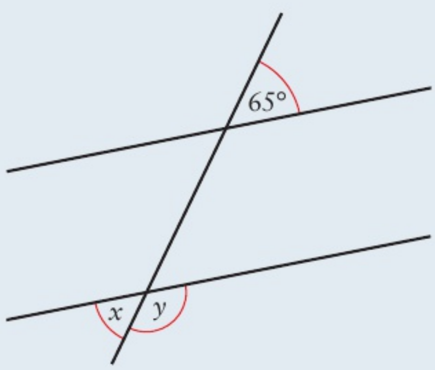
a



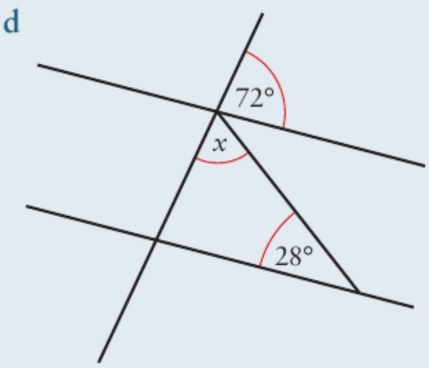
b



c

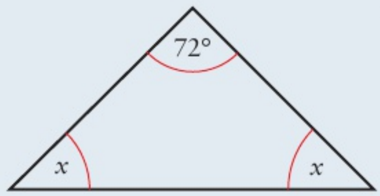


d

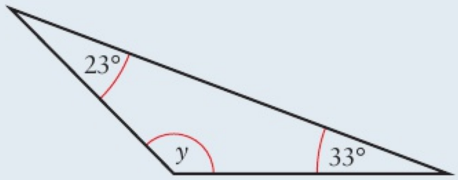


3 Find the value of the variable in each case.

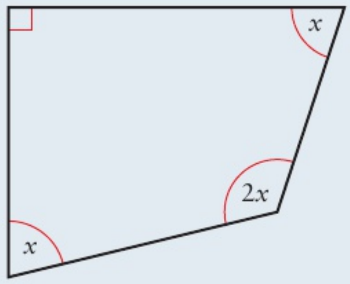
a



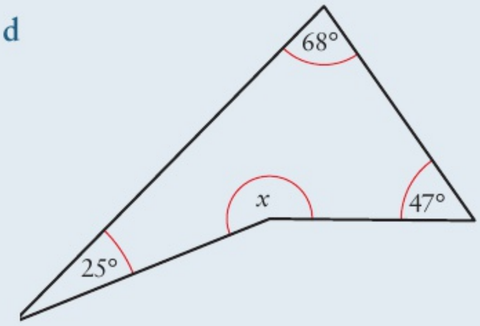
b



c

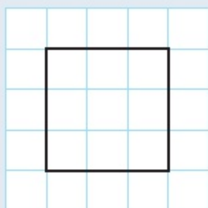


d

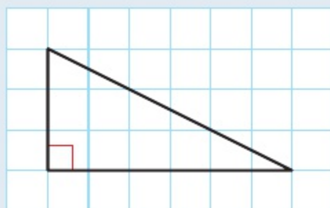


4 Calculate the perimeter of each shape.

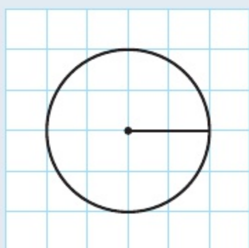
a



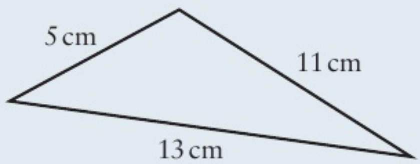
b



c

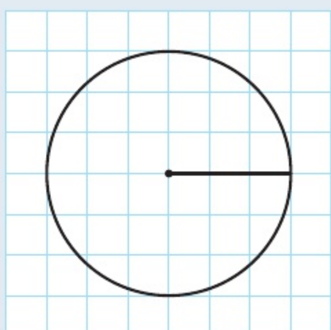


d

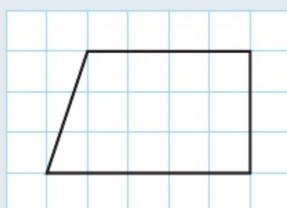


5 Find the area of each shape drawn on grid paper.

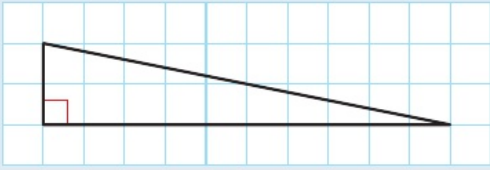
a



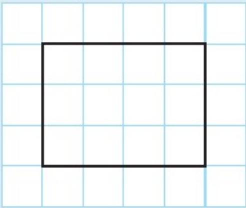
b



c

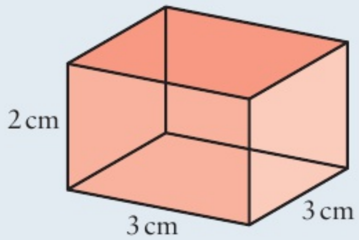


d



6 Find the surface area of this 3D shape.

Challenge Q6



1.4 Probability and statistics

1.4.1 Basic probability

Explore 1.11

a Can you determine the probability of each of the following events?

i Getting tails when you flip a coin.

ii Rolling a 7 when you roll a six-sided dice.

iii It will rain tomorrow.

iv Choosing a weekday that starts with the letter Z.

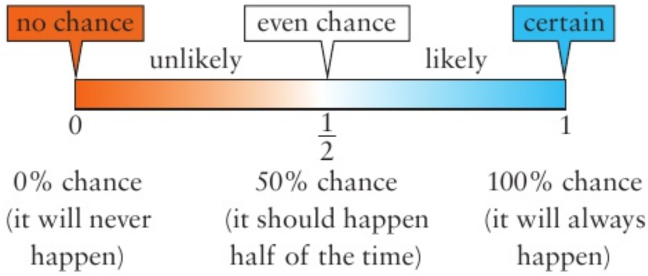
b Can a probability be greater than 1?

- A **simple event** is an event that can only happen in one way – in other words, it has a single outcome. For example, the event that a coin shows tails when flipped once.
- A **sample space** is the list of all possible outcomes for an event.
- If all outcomes are equally likely, the probability (P) of an event (X) can be represented by following formula:

  Probability of an event $P(X) = \frac{n(X)}{n(S)}$ where S represents the sample space.
- All probabilities are between 0 (impossible) or 0% and 1 (certain) or 100%. See the probability line on the next page.

Reminder

$n(X)$ is the number of outcomes that match the event X, and $n(S)$ is the total number of possible outcomes.



- Probabilities can be written as fractions, percentages or in decimal form. For example, 50%, 0.5 and $\frac{1}{2}$ all represent the same probability.

Worked example 1.19

You throw a regular dice. Calculate the probability that the number facing up is:

a 1 **b** an even number **c** a number greater than 4

d 7 **e** a factor of 6.

Solution

The sample space of the events is $S = \{1, 2, 3, 4, 5, 6\}$. Use the probability formula to calculate each probability.

a $P(1) = \frac{1}{6}$ **b** $P(\text{an even number}) = \frac{3}{6} = \frac{1}{2}$

c $P(5 \text{ or } 6) = \frac{2}{6} = \frac{1}{3}$ **d** $P(7) = \frac{0}{6} = 0$

e $P(1, 2, 3, 6) = \frac{4}{6} = \frac{2}{3}$

Look back

a Since there is only one 1 in the sample space, the probability of throwing 1 is $\frac{1}{6}$

b There are three possible even numbers, 2, 4 and 6. Thus, the probability of throwing an even number is $\frac{3}{6} = \frac{1}{2}$

c There are two numbers (5 and 6) greater than 4, so the probability is $\frac{2}{6} = \frac{1}{3}$

d Since there is no 7 in the sample space, the probability of throwing a 7 is 0, i.e. impossible.

e The factors of 6 are 1, 2, 3 and 6. Therefore the probability of throwing a factor of 6 is $\frac{4}{6} = \frac{2}{3}$

Hint

The event of getting 1 is a simple event while the event of getting an even number is not.

Worked example 1.20

There are 3 red, 4 blue and 5 green balls in a bag. If a ball is randomly picked from the bag, find the probability of choosing:

a a red ball **b** a blue ball **c** a green ball.

Solution

We need to find the probability of each event.

The sample space is $3 + 4 + 5 = 12$. There are 3 red, 4 blue and 5 green balls. The probability of choosing each colour can be calculated by using the formula.

a $P(\text{red ball}) = \frac{3}{12} = \frac{1}{4}$ **b** $P(\text{blue ball}) = \frac{4}{12} = \frac{1}{3}$ **c** $P(\text{green ball}) = \frac{5}{12}$

If we find the sum of these probabilities, it gives us the total probability of 1, which is true.

Fact

The sum of the probabilities of all possible events is 1 or 100%.
If there is an event A in a sample space S, the probability of A happening is $P(A)$
The probability of event A **not** happening is $1 - P(A)$ or $P(A')$.
A' is called the **complement of** A.

Worked example 1.21

Write the complement of each event.

a Rolling a dice and getting a 1 **b** Flipping a coin and not getting heads

c It will rain at 12:00 **d** You will win a dance contest

Solution

a Not rolling 1, or rolling 2, 3, 4, 5 or 6 **b** Getting heads

c It won't rain at 12:00 **d** You will not win a dance contest

1.4.2 Statistics

Explore 1.12

Use the number set below for questions a and b.

2 3 3 3 4 4 4 4 5 5 5 6 6 7 8 8 9 9 10 10 10

a Can you determine the mean, mode and median of the number set?

b Do you know what a frequency chart is? Can you make a frequency chart for the number set?

Statistics deals with collecting, organising and evaluating a data set based on the population (**census**) that you are interested in, or on a randomly selected part of the population (**sample**).

The data set in Explore 1.12 can be organised in a table where it is easy to see repeated data. This **frequency chart** shows the same data set.

Outcome	Tally	Frequency
2	|	1
3	|||	3
4	||||	4
5	|||	3
6	||	2
7	|	1
8	||	2
9	||	2
10	|||	3

- The **Outcome** column lists the possible 'values' of data points.
- The **Tally** column is a recording of 'the occurrence' of each data point, one entry each time.
- The **Frequency** column represents the number of times each outcome occurs.

When the data set is organised, it is easier to analyse. The following types of measurement can be determined.

- The **range** shows the spread of the data set. The range is calculated by taking the difference between the highest and lowest data values.
- The **mean** is the average of the data. In order to calculate the mean, add all the data values and then divide by the number of data values. The mean is usually denoted by $\bar{x}$.
- The **mode** is the data value that has the highest frequency, or the data value that occurs the most in the data set. If there are two modes in a data set, it is called **bimodal**.
- The **median** is the middle value when the number set is arranged in order. If there are two middle numbers, the median is the average of the two middle values.

  The mean, mode and median are called **measures of central tendency**.

Worked example 1.22

For the data set below:

2 3 4 4 2 1 3 5 6 6 2 2 4 5 5 3 1 2 2 4 5 6 1 4 6

a make a frequency chart of the data set

b calculate the range, mean, mode and median.

Solution

Understand the problem

We need to create a frequency chart and calculate the range, mean, mode and median.

Make a plan

a Create a frequency table with tally and frequency columns.

b Use the definition of the range and central tendencies to calculate each of them.

Carry out the plan

a There are six outcomes. The tally and frequency columns show how many times each outcome occurs.

Outcome	Tally	Frequency
1	|||	3
2	|||||	6
3	|||	3
4	||||	5
5	||||	4
6	||||	4
	Total	**25**

b Range = 6 − 1 = 5

Mean = $(3 \times 1 + 6 \times 2 + 3 \times 3 + 5 \times 4 + 4 \times 5 + 4 \times 6) \div 25 = 3.52$

Mode = 2

Median is the middle number, which is the 13th data value, which is 4.

The frequency table make sense since the outcomes are the numbers from 1 to 6. The tally and frequency chart determines how many times they are each repeated.

The mode is the most repeated data value, which is 2.

The mean is the average of the data set. The sum of the data values is 88 and there are 25 values, so the mean is $\frac{88}{25} = 3.52$

The median is the middle number when the values are sorted into numerical order. Since there are 25 numbers, the middle one is the 13th number, which is 3.

Fact

The **outliers** are data values that are not consistent with the rest of the data, because they are either too small or too big. For example, 35 is an outlier for the data set:
2 2 3 4 5 5 7 7
8 1 9 8 10 10 **35**

Investigation 1.2

Exploring the number pattern

Investigate the following questions for the 30 numbers in the data set below.

12 13 12 14 15 12 13 15 16 12 14 13 13 16 17
18 18 19 12 13 14 19 18 17 20 19 17 11 12 20

1. Create a frequency chart for the data set.
2. Find the mean, mode and median of the data set.
3. Add an outlier of 88 to the data set.
4. Find the new mean, mode and median.
5. Which of the central tendency measurements is most affected by the outlier?
6. Can you generalise your finding and make a statement about how an outlier affects the central tendency? Justify your statement.

Practice questions 1.4

1. List the sample space for these events.
   - a A coin is flipped
   - b A dice is rolled
   - c The months that start with the letter M
   - d The days of the week that start with the letter W
2. A letter is randomly chosen from the letters of the word ‘algebra’. Find the probability of choosing:
   - a the letter ‘a’
   - b a vowel
   - c the letter ‘k’
   - d a letter that is not ‘b’.
3. Find the probability of the complementary event in each case.
   - a The probability of event A is 40%
   - b The probability of getting a 1 when you roll a dice
   - c The probability of not getting tails when you flip a coin
   - d The probability of winning a game is 0.8

Challenge Q4

4 When you flip a coin three times, there are eight possible outcomes. For example, one outcome is: head, head, head (HHH)

a Write down the sample space.

b What is the probability of getting two tails?

c What is the probability of getting no heads?

d What is the probability of having at least one tail?

5 a Create a frequency chart for this data set:

0 0 2 1 3 1 2 4 5
5 2 0 0 2 1 5 6 8

b What is the sum of the data values?

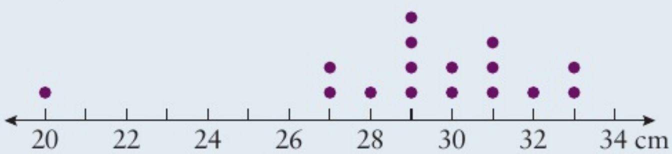
c What is the mode?

d What is the mean?

e What is the range?

f What is the median?

6 The lengths of 16 fish were measured. The results are shown on this dot plot.



a Find:

i the mode ii the median iii the range.

b Which measurement is the outlier?

Self assessment

- I can classify numbers.
- I can distinguish between rational and irrational numbers.
- I can convert recurring decimals into fractions.
- I know what prime and composite numbers are.
- I can write a composite number as the product of prime numbers.
- I can describe square and cube numbers.
- I know the divisibility rules.
- I can find the HCF and the LCM of number pairs.
- I know the order of operations.
- I know what linear patterns are.
- I can find the rule of a linear pattern.
- I can operate on algebraic expressions.
- I know inverse operations.
- I can solve one-step and two-step equations.
- I can describe inequality signs.

1 Year 2 review

- I can solve inequalities and show the solution on a number line.
- I can identify adjacent angles.
- I can identify supplementary and straight angles.
- I can identify complementary angles and right angles.
- I can identify vertically opposite, corresponding and alternate angles.
- I know that the sum of the interior angles of a triangle is 180 degrees.
- I know that the sum of the interior angles of a quadrilateral is 360 degrees.
- I can find the perimeter of a shape.
- I can find the circumference of a circle.
- I can find the areas of a triangle, square, rectangle and circle.
- I can find the area of composite shapes.
- I can find the probability of an event.
- I know that probability is between 0 and 1.
- I know what complementary events are.
- I can describe mean, mode and median.
- I can create a frequency table.

Check your knowledge questions

1 a Write down the first 12 multiples of the number 11.

b Write down all the factors of 24.

c Calculate the highest common factor (HCF) of 45 and 60.

d Write down the least common multiple (LCM) of 12 and 11.

2 a Write down all the prime numbers between 10 and 40.

b Write down all composite numbers less than 15.

c Write 240 as the product of prime numbers.

3 Organise these numbers by type in the table.

Number	Natural	Integer	Rational	Irrational	Real
79					
-11					
3π					
3.5					
$\frac{-1}{5}$					
$(-2)^4$					
$\sqrt{16}$					
$\sqrt{15}$					
$(2.3)^2$					
3.222…					

4 Use the order of operations to perform these calculations.

a $-8 - 4 \div 2 \times 3$

b $\sqrt{25} - 3^3 \div 3 + 4$

c $(3 \times 5 - 2^2) - 13 + \frac{4}{2}$

5 Find the fraction that represents the recurring decimal 12.121212…

6 Complete the table below for the equation $y = 3x + 7$

x	0	−1	1	2
y				

7 Write down the rule for this pattern.

x	0	1	2	3
y	−5	0	5	10

8 Simplify each expression.

a $2x - 3y + x - 11y$

b $\frac{x}{2} - \frac{x}{3}$

c $16ab^2 \div 8a^2b$

d $\frac{-25x \times 6y}{10xy}$

9 Solve these equations.

a $\frac{x}{3} - 5 = 11$

b $2(x - 1) + 3(1 - x) = -1$

c $5(x + 3) = 2(3x - 1)$

d $\frac{4x + 5}{3} = 3$

10 Solve these inequalities and graph each solution on a number line.

a $2x - 5 \geq 11$

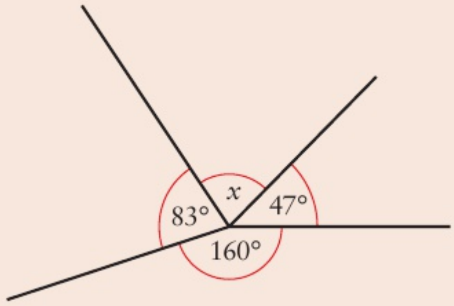
b $\frac{y}{2} - 7 \leq 3$

c $2(z - 3) < z + 1$

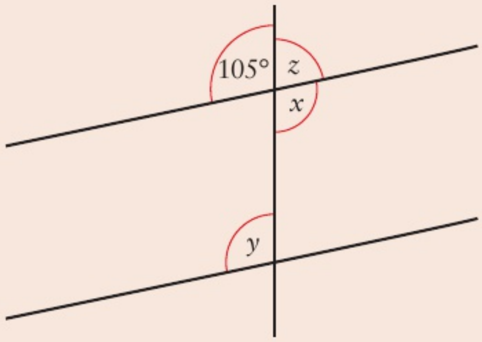
d $\frac{w}{3} + \frac{w}{2} > 5$

11 Find the values of the missing angles.

a



b



In part b, the two diagonal lines are parallel.

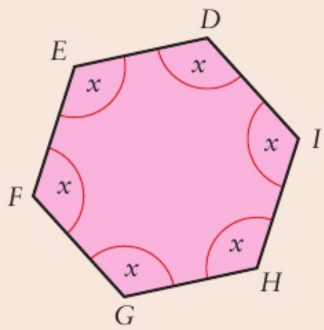
c

Challenge Q11d

Hint Q11d

DEFGHI is a regular hexagon.

d



Challenge Q12

12 If the value of the circumference of a circle is double the value of its area, calculate the radius.

13 In a bag, there are 5 blue pens, 4 red pens and 3 black pens. Find the probability of randomly choosing:

a a pen that is red
b a pen that is black
c a pen that is not blue
d a pen that is yellow.

14 For the data set shown:

12 13 14 14 12 11 13 15 16 16 12 12 14
15 15 13 11 12 12 14 15 16 11 14 16

a draw a frequency chart

b calculate the range, mean, median and mode.

Year 2 extension

2

2 Year 2 extension

KEY CONCEPT

Form

RELATED CONCEPTS

Equivalence, Representation, Simplification, Systems

GLOBAL CONTEXT

Scientific and technical innovation

Statement of inquiry

Representing a quantity in equivalent forms allows us to understand relationships between scientific variables.

Factual

- What is a ratio?
- How can two ratios be equivalent?

Conceptual

- Can you express ratio in different forms?
- How do you split a number into proportions?

Debatable

- Which form of a ratio is more applicable: fraction, decimal or percentage?
- Can you have the ratio of different units? What does that mean?

Do you recall?

1 Can you express the following statements as percentages? How?
   a The number of boys to girls in a class.
   b 500 g to 2 kg.
   c 20% of the students in a school play an instrument.

2 Show that $\frac{12}{15} = \frac{56}{70} = 75\%$ is not true. Explain your logic.

2.1 Percentage

2.1.1 Revision of percentages

A percentage describes a quantity as a fraction of one hundred.

So, 45% means $\frac{45}{100}$, a% means $\frac{a}{100}$, and so on.

Explore 2.1

Can you write the following numbers as percentages?

$\frac{75}{100}$ 75 $\frac{1}{2}$ 0.34   1.25 $\frac{17}{50}$

Can you identify which two numbers have the same percentage equivalents?

Worked example 2.1

Write each of these fractions as a percentage.

$\frac{91}{100}$ $\frac{3}{4}$ $\frac{3}{10}$ $\frac{12}{25}$ $\frac{120}{150}$

Solution

We can make sure each fraction has 100 in the denominator by simplifying or expanding the fraction.

$$\frac{91}{100} = 91\%$$

$$\frac{3}{4} = \frac{3 \times 25}{4 \times 25} = \frac{75}{100} = 75\%$$

$$\frac{3}{10} = \frac{3 \times 10}{10 \times 10} = \frac{30}{100} = 30\%$$

$$\frac{12}{25} = \frac{12 \times 4}{25 \times 4} = \frac{48}{100} = 48\%$$

$$\frac{120}{150} = \frac{120 \div 3}{150 \div 3} = \frac{40}{50} = \frac{40 \times 2}{50 \times 2} = \frac{80}{100} = 80\%$$

Looking back at the solution, we could have done it differently by expressing each fraction as a decimal first and then changing it into a percentage. For example: $\frac{12}{25} = 0.48 = 48\%$

Hint

All terminating and recurring decimals can be converted into fractions and percentages.

For example, 0.25 equals $\frac{25}{100}$ or 25%

Worked example 2.2

Convert each of these decimals into a percentage.

0.75   0.01   3.6   0.1234

Solution

Write each decimal as a fraction with denominator 100.

$0.75 = \frac{75}{100} = 75\%$

$0.01 = \frac{1}{100} = 1\%$

$3.6 = \frac{360}{100} = 360\%$

$0.1234 = \frac{12.34}{100} = 12.34\%$

Looking back, we could have chosen to multiply each decimal by 100%
For example, $3.6 \times 100\% = 360\%$

Worked example 2.3

Arrange these numbers in ascending order.

$\frac{4}{5}$ $\quad 66\%$ $\quad 0.49$ $\quad 1\frac{1}{10}$ $\quad \frac{8}{7}$

Solution

Understand the problem

We need to write all numbers in the same format, to put them in the correct order.

Make a plan

One way is to convert all numbers to percentages.

Carry out the plan

$\frac{4}{5} = \frac{4 \times 20}{5 \times 20} = \frac{80}{100} = 80\%$

66%

$0.49 = \frac{49}{100} = 49\%$

$1\frac{1}{10} = \frac{11}{10} = \frac{11 \times 10}{10 \times 10} = \frac{110}{100} = 110\%$

$\frac{8}{7} \approx 1.143 = \frac{114.3}{100} = 114.3\%$

$49\% < 66\% < 80\% < 110\% < 114.3\%$

Thus, the numbers in ascending order are: $0.49 \quad 66\% \quad \frac{4}{5} \quad 1\frac{1}{10} \quad \frac{8}{7}$

Look back

We could have chosen to convert all the numbers into decimals instead.

$\frac{4}{5} = 0.8$

$66\% = 0.66$

0.49

$1\frac{1}{10} = 1.1$

$\frac{8}{7} \approx 1.14$

$0.49 < 0.66 < 0.8 < 1.1 < 1.14$

Thus, we get the same result as above:

$0.49 \quad 66\% \quad \frac{4}{5} \quad 1\frac{1}{10} \quad \frac{8}{7}$

Reflect

Can you think of another way of ordering the numbers?

Choose an appropriate method to arrange the numbers below in ascending order. Why do you prefer the method you chose?

$\frac{7}{8} \quad 58\% \quad \frac{10}{9} \quad 0.7 \quad 1\frac{12}{20}$

2.1.2 Percentage of a number

Explore 2.2

Is 10% of \$20 the same as 20% of \$10? Try the same comparison with 60% of \$84 and 84% of \$60. Can you explain the reason for these answers?

Explore 2.3

You would like to purchase a €550 smartphone, which is on sale with a 20% discount. A newer version of the same phone will be offered at €425 in a month's time. Should you buy now or wait?

Hint

To find a percentage of a number, write the percentage as a decimal or fraction and multiply by the number.

To find what percentage a number is as part of a whole, write the number as a fraction of the whole and **multiply by 100%**

Worked example 2.4

Find the percentage of each of these numbers.

a 20% of 50 **b** 50% of 20

Solution

a $20\% \text{ of } 50 = \frac{20}{100} \times 50 = \frac{1000}{100} = 10$

b $50\% \text{ of } 20 = \frac{50}{100} \times 20 = \frac{1000}{100} = 10$

Investigation 2.1

It is a good practice to check your receipt after making purchases. Sometimes there may be mistakes.

You recently went shopping. Here is your receipt. Note that some items were discounted.

Item type	Price
Washing powder	€10.00
Discount 1	−€2.50
Carrots	€2.50
Olive oil	€4.00
Apples	€1.50
Chicken	€7.20
Discount 2	−€3.00
Fish	€6.00
Pasta	€3.50
Discount 3	−€1.75
Mixed vegetables	€5.75
Granola	€4.25
Bananas	€1.00

a How much have you spent in total?

b How much discount have you received in total?

c What percentage of items are discounted?

d What percentage of items are not discounted?

Worked example 2.5

Suhrit lives in London and has a monthly salary of £3200. His friend Elias lives in Berlin and has a monthly salary of €2750. They are comparing their monthly expenditure.

a Find Suhrit and Elias' individual expenditures as a percentage of their salaries. Hence work out the overall percentage expenditure for each of Elias and Suhrit.

b Whose total expenditure represents a greater proportion of their salary? Explain your reasoning.

c Given that €1.15 = £1.00, who saves more money? Why?

Suhrit's expenditure:

Item	Amount spent
Rent	£1350
Transport	£250
Food	£450
Clothes	£200
Other	£550

Elias' expenditure:

Item	Amount spent
Rent	€950
Transport	€350
Food	€450
Clothes	€250
Other	€400

Solution

a Suhrit's expenditure:

Item	Amount spent	Percentage (1 d.p.)
Rent	£1350	$\frac{1350}{3200} \times 100\% = 42.2\%$
Transport	£250	$\frac{250}{3200} \times 100\% = 7.8\%$
Food	£450	$\frac{450}{3200} \times 100\% = 14.1\%$
Clothes	£200	$\frac{200}{3200} \times 100\% = 6.2\%$
Others	£550	$\frac{550}{3200} \times 100\% = 17.2\%$
TOTAL	**£2800**	**87.5%**

Elias' expenditure:

Item	Amount spent	Percentage (1 d.p.)
Rent	€950	$\frac{950}{2750} \times 100\% = 34.5\%$
Transport	€350	$\frac{350}{2750} \times 100\% = 12.7\%$
Food	€450	$\frac{450}{2750} \times 100\% = 16.4\%$
Clothes	€250	$\frac{250}{2750} \times 100\% = 9.1\%$
Others	€400	$\frac{400}{2750} \times 100\% = 14.6\%$
TOTAL	**€2400**	**87.3%**

b Suhrit spends a total of 97.5% of his salary and Elias spends a total of 97.3% of his salary. Therefore Suhrit's expenditure is closest to his salary.

c Suhrit saves: $3200 - 2800 = £400$

Elias saves: $2750 - 2400 = €350$

We are given that €1.15 = £1.00 so 1 pound sterling is worth more than 1 Euro. Therefore, Suhrit saves more money.

2.1.3 Application of percentages

Explore 2.4

Kaan got a 10% discount on a mountain bike that originally cost $180. Becky said, 'I got only a 5% discount on the bike I bought, but I still paid less than Kaan paid.' Can you explain how this could be true?

Explore 2.5

Five years ago, Lisa and Katashi bought a house for €250,000. They want to move to a different area and need to sell this house. If the market price of houses in this area increased by 15% during the five years, how much do you think they should try to sell their house for?

Increase and **decrease** help us to find the new amount of a quantity.

To decrease by a percentage, subtract the percentage decrease from 100% and find the new amount.

For example, what is the actual sale price for an item that costs €280 if the discount rate is 20%?

The sale price in this case is: 100% − 20% = 80%
Thus the discounted price of the item is: 80% × 280 = €224

To increase by a percentage, add the percentage increase to 100% and find the new amount.

For example, to earn a profit on sales, a shopkeeper increases his cost price by 20%. What is the selling price for an item that costs €280?

The selling price as a percent is 100% + 20% = 120%

Thus, the actual selling price is 120% × 280 = €336

Reflect

Can you think of another approach to find the percentage decrease or increase in the above examples?

Worked example 2.6

Eugene bought a car for €6000 and sold it to Pune at a loss of 75% of its value, because of damage incurred as a result of an accident. Pune got the car repaired and sold it to Melissa for 250% more than he paid for it. How much did Melissa pay for the car? Explain your solution.

Solution

First we find out how much Pune paid for the car with a 75% discount on the original price:

Pune paid 100% − 75% = 25% of €6000

$\frac{25}{100} \times 6000 = €1500$

Then we find how much Melissa paid for the car with a 250% increase on how much Pune had paid:

Pune sold the car at a price of 100% + 250% = 350% of €1500

Therefore, Melissa paid $\frac{350}{100} \times 1500 = €5250$

Looking back, 25% of €6000 is €1500, which is how much Pune paid for the car.

350% of €1500 is €5250, which is how much Melissa paid for it.

Practice questions 2.1

1 Convert each of these fractions into a percentage.

a $\frac{3}{4}$ b $\frac{12}{20}$ c $\frac{125}{100}$ d $1\frac{11}{25}$

2 Convert each of these percentages into a fraction in its simplest form.

a 34% b 15.4% c 111% d 5%

3 Change each of these decimals into a percentage.

a 0.23 b 12.5 c 0.5 d 0.01

4 Find the percentage of each of these quantities.

a 5% of 1200 kg b 25% of 100 m

c 15% of 60 min d 1.1% of 11

5 a Increase €25 by 25% b Increase 150 m by 30%

c Decrease 2000 mm by 40% d Decrease 40 hours by 5%

6 The average cost of a desktop PC in 2005 was $805. In 2015, this went down to $544.

What percentage decrease is this?

7 In 2015, the annual average daily traffic (AADT) figure for the A44 road west of Chipping Norton, England, was 6500 vehicles. By 2019, this figure had dropped by 9%, thanks to traffic calming measures and new roads. What was the AADT figure for 2019?

8 If shares that were bought for €2,000,000 are sold at a profit of 8%, how much would the profit be?

9 The table on the next page lists the top 10 countries for solar power capacity in 2019, as well as the total for all other countries. Calculate the percentage of the total solar power capacity for:

a China b Germany

c All other countries d Australia.

Country	Solar power capacity (Megawatts)
China	204 700
United States	75 900
Japan	63 000
Germany	49 200
India	42 800
Italy	20 800
Australia	14 600
United Kingdom	13 300
South Korea	11 200
France	9 900
All other countries	121 600

2.2 Ratio, proportion and rates

2.2.1 Ratio basics

Explore 2.6

a Do the following numbers represent the same value?

$1:4 \quad 25\% \quad 0.25 \quad \frac{7}{28}$

b If you split €80 between two people in the ratio 2 : 3, how much would each of them receive? Why?

The comparison of two numbers is called a **ratio**. Ratios can be written in the form $\boldsymbol{a:b}$ or $\frac{a}{b}$ or $\boldsymbol{a}$ **to** $\boldsymbol{b}$. The terms in a ratio should be in the same units. For example, 1 litre of juice to 3 litres of water can be represented as $1:3$ or $\frac{1}{3}$

Worked example 2.7

Neva buys a book for €25 and later sells it for €15. Find the ratio of her:

a selling price to cost price

b cost price to selling price to loss amount.

Solution

a selling price to cost price: $\frac{15}{25} = \frac{3}{5}$ or 3 : 5

b cost price to selling price to loss amount is 25 : 15 : 10 or 5 : 3 : 2

Worked example 2.8

There are 15 boys and 18 girls in a class. Find the ratio of:

a boys to girls

b girls to whole class.

Solution

There are 15 + 18 = 33 students in the class. Thus, the ratios of:

a boys to girls is 15 : 18 or 5 : 6

b girls to whole class is 18 : 33 or 6 : 11

Hint

If the units of the terms of a ratio are different, then convert them to the same unit.

Worked example 2.9

Find the following ratios.

a 5 cm to 1 m

b 50 seconds to 2 minutes

c 5 kg to 2000 g

d 500 millilitres to 2 litres

Solution

In each case we first convert to the same unit.

a 5 cm to 1 m (100 cm) is 5 : 100 or 1 : 20

b 50 seconds to 2 minutes (120 seconds) is 50 : 120 or 5 : 12

c 5 kg to 2000 g (2 kg) is 5 to 2 or 5 : 2

d 500 millilitres to 2 litres (2000 millilitres) is 500 : 2000 or 1 : 4

Worked example 2.10

If two people share €150 in the ratio of 2 : 3, how much would each of them receive? Show your solution.

Solution

We need to share €150 in the ratio of 2 : 3. That is, we have 5 shares altogether.

One person gets $\frac{2}{5}$ of €150 and the other person gets $\frac{3}{5}$ of €150

First person: $\frac{2}{5}$ of €150 is $\frac{2}{5} \times 150 = \frac{300}{5} = €60$

Second person: $\frac{3}{5}$ of €150 is $\frac{3}{5} \times 150 = \frac{450}{5} = €90$

Looking back, we can show that our answer is correct since

$€60 + €90 = €150$ and $\frac{60}{90} = \frac{2}{3}$

Practice questions 2.2.1

1 Mako has 4 blue ribbons and 6 red ribbons. Find the ratio of:
- a blue ribbons to red ribbons
- b red ribbons to blue ribbons
- c blue ribbons to total number of ribbons
- d red ribbons to total number of ribbons.

2 A school has 120 girls and 150 boys. Find the ratio of:
- a boys to girls
- b girls to boys
- c boys to total number of students
- d girls to total number of students.

3 If Tom and Clara share €270 in the ratio of 2 : 3, what is the difference between their shares?

4 Find the ratio of:
- a 10 centimetres to 25 millimetres
- b 5 minutes to 100 seconds
- c 1 kilogram to 100 grams
- d 5 litres to 25 centilitres.

Challenge Q5

5 A standard tennis court is 78 feet long by 36 feet wide. A basketball court is 94 feet long by 58 feet wide.

a Find the ratio of the perimeter of the tennis court to the perimeter of the basketball court.

b Find the ratio of the area of the tennis court to the area of the basketball court.

c Which of the two ratios is bigger?

6 The road between Balladonia and Caiguna in Western Australia stretches for about 147 km without a single bend. Along the road, there are only two places to stop: the Baxter rest area and the Caiguna Blowhole.

Route	Distance
A: Balladonia to the Baxter rest area	88 km
B: Baxter rest area to the Caiguna Blowhole	53 km
C: The Caiguna Blowhole to Caiguna	6 km

Using these distances, find the ratio of:

a the distance between Balladonia and Baxter to the distance between Baxter and Caiguna Blowhole

b the distance between Baxter and Caiguna Blowhole to the distance between Caiguna Blowhole and Caiguna

c the distance between Balladonia and Caiguna Blowhole to the distance between Balladonia and Caiguna

d the distance between Balladonia and Caiguna Blowhole to the distance between Balladonia and Caiguna.

Challenge Q7

7 The ratio of interior angles of a triangle is $2:3:4$. Eduardo says this is a right-angled triangle and Aiswarya says it is not a right-angled triangle. Who is correct? Why?

8 Simplify each of these ratios.

a $12:20$ **b** $0.25:1.75$ **c** $\frac{32}{68}$ **d** 60%

9 There are 50 stations on the Northern Line (NL) on London's underground network. On the Metropolitan Line (ML) there are 34 stations, and on the Bakerloo Line (BL) there are 25 stations. Calculate the ratio of stations on:

a NL to ML **b** NL to BL **c** BL to ML **d** BL to NL

2.2.2 Equivalent ratios

Explore 2.7

a Dylan has 300 friends on social media and Cameron has 375. April has 400 friends, while Yvette has 600 friends. Are the ratios of Dylan's friends to Cameron's and April's to Yvette's the same?

b Lydia travels 400 km in 5 hours. If she drives with the same speed, how long will it take to travel another 660 km?

If two ratios are equivalent, we say that they are in **proportion**.

For example, $\frac{1}{2}$ and $\frac{8}{16}$ are in proportion because both ratios have the same value of 0.5

If two ratios are equal, such as $\frac{a}{b} = \frac{c}{d}$, then the **cross-product** is also equal: $a \times d = b \times c$

For example, if $\frac{3}{8} = \frac{9}{24}$, then $3 \times 24 = 8 \times 9, 72 = 72$

Worked example 2.11

Find the missing value in each equivalent ratio.

a $\frac{x}{15} = \frac{55}{75}$

b $5:11$ is equivalent to $25:y$

c $121:55 = x:30$

d $1.2:2.7 = 16:z$

Solution

Understand the problem

We need to find the value of the variable so that each equivalence is true.

Make a plan

Set up the cross-product for each ratio and solve for the unknown variable.

Carry out the plan

a $\frac{x}{15} = \frac{55}{75}$

$75 \times x = 55 \times 15$

$x = \frac{55 \times 15}{75} = 11$

b $\frac{5}{11} = \frac{25}{y}$

$5 \times y = 11 \times 25$

$y = \frac{11 \times 25}{5} = 55$

c $\frac{121}{55} = \frac{x}{30}$

$55 \times x = 121 \times 30$

$x = \frac{121 \times 30}{55} = 66$

d $\frac{1.2}{2.7} = \frac{16}{z}$

$1.2 \times z = 16 \times 2.7$

$z = \frac{16 \times 2.7}{1.2} = 36$

Look back

If the solutions are written as equivalent ratios and simplified, it is clear that the ratios are equal and the solutions are correct.

a $\frac{11}{15} = \frac{55 \div 5}{75 \div 5}$

b $\frac{5}{11} = \frac{25 \div 5}{55 \div 5}$

c $\frac{121}{55} = \frac{66}{30}$

$\frac{121 \div 11}{55 \div 11} = \frac{66 \div 6}{30 \div 6} = \frac{11}{5}$

d $\frac{1.2}{2.7} = \frac{16}{36}$

$\frac{1.2 \times 10 \div 3}{2.7 \times 10 \div 3} = \frac{16 \div 4}{36 \div 4} = \frac{4}{9}$

Worked example 2.12

In 2019, the ratio of wind energy production of Denmark to the United Kingdom was 12 : 5. The figures are based on the percentage of the total energy production that comes from wind energy. If Denmark produced about 48% of its energy from wind power, then how much of the UK's energy was from wind power?

Solution

Understand the problem

We need to find the UK's wind energy production in 2019.

Make a plan

Set up the proportion and solve for the unknown variable:

$$\frac{12}{5} = \frac{48}{w}$$

where w represents wind energy production of UK.

Carry out the plan

Let's solve the equivalent ratio:

$$\frac{12}{5} = \frac{48}{w}$$

$$12 \times w = 5 \times 48$$

$$w = \frac{240}{12}$$

$$= 20$$

Therefore, the UK's wind energy production was 20%

Look back

Simplifying the ratio of 48% : 20% gives 12 : 5, which is the correct ratio of wind power production in Denmark and the UK.

Investigation 2.2

The table below is a summary of the x- and y-coordinates of a function. Use this table to answer the following questions.

x	y
$x_1 = 1$	$y_1 = 3$
$x_2 = 2$	$y_2 = 6$
$x_3 = 3$	$y_3 = 9$
$x_4 = 4$	$y_4 = 12$
$x_5 = \dots$	$y_5 = \dots$
$x_6 = \dots$	$y_6 = \dots$
$x_7 = \dots$	$y_7 = \dots$

1 Find the missing x and y values in the table.

2 Complete the table below by using the values from the first table. The first two rows have been done for you.

$x_n - x_{n-1}$	$y_n - y_{n-1}$
$x_2 - x_1 = 2 - 1 = 1$	$y_2 - y_1 = 6 - 3 = 3$
$x_3 - x_2 = 3 - 2 = 1$	$y_3 - y_2 = 9 - 6 = 3$
$x_4 - x_3 = \dots$	$y_4 - y_3 = \dots$
$x_5 - x_4 = \dots$	$y_5 - y_4 = \dots$
$x_6 - x_5 = \dots$	$y_6 - y_5 = \dots$
$x_7 - x_6 = \dots$	$y_7 - y_6 = \dots$

3 What do you notice about the x and y differences? Is there a pattern?

4 If you find the ratio of $y_n - y_{n-1}$ to $x_n - x_{n-1}$ values, are they proportional? Explain your answer.

5 Is there a constant for this difference? Justify your answer.

Hint Q2

$x_n - x_{n-1}$ and $y_n - y_{n-1}$ means taking the difference of two consecutive x and y values.

Fact

If the ratio of the differences in the x and y values are equal, then we call this relationship a **linear pattern**. Linear patterns are represented as **straight lines** in the coordinate system.

Worked example 2.13

Look at these x and y relationships and state whether they are proportional/linear or not.

a

x	y
1	2
2	4
3	6
4	8
5	10
6	12

b

x	y
1	3
2	6
3	9
4	12
7	21
12	36

Solution

We need to check the two tables to see if they represent a linear relationship. They will form a linear relationship if the ratio of the x and y differences is constant.

Set up the ratios of x and y differences and check whether they are proportional.

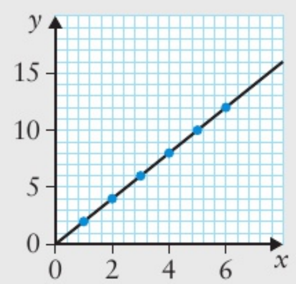
a $\frac{4-2}{2-1} = \frac{6-4}{3-2} = \frac{8-6}{4-3} = \frac{10-8}{5-4} = \frac{12-10}{6-5} = 2$

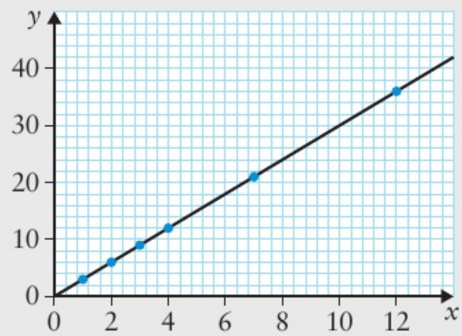
So, this is a proportional and linear relationship.

b $\frac{6-3}{2-1} = \frac{9-6}{3-2} = \frac{12-9}{4-3} = \frac{21-12}{7-4} = \frac{36-21}{12-7} = 3$

So, this is a proportional relationship.

Looking back, if you plot the values as points on a coordinate grid, both resulting graphs are straight lines passing through the origin.





Connections

Linear relationships can be described as straight lines in a coordinate plane. Proportional relationships can be described as straight lines that pass through the origin. You will explore linear relationships and their graphs in Chapter 5: Coordinate geometry.

Fact

Scale drawings are one of the most useful applications of equivalent ratios. Scale drawings are given a scale factor, such as 1 : 1000. This means that every 1 unit in the drawing is equivalent to 1000 units in the real object.

Worked example 2.14

Simplify each of these scales.

a 1 mm : 10 m **b** 2 cm : 1 km **c** 50 seconds : 20 hours

Solution

a $\frac{1\text{ mm}}{10\text{ m}} = \frac{1\text{ mm}}{10\,000\text{ mm}} = 1:10\,000$ **b** $\frac{2\text{ cm}}{1\text{ km}} = \frac{2\text{ cm}}{100\,000\text{ cm}} = 1:50\,000$

c $\frac{50\text{ s}}{20\text{ h}} = \frac{50\text{ s}}{1200\text{ s}} = 1:24$

Fact

At the Museum of Natural History in Oxford, on one side of the hall is a scale model of the Sun, about the size of a medium watermelon. If you walk to the opposite side of the upper galleries, you will find the Earth to scale, about the size of a dried pea. The Moon, a few inches away, has a diameter of about half of a cumin seed. To scale, the nearest star to us, Alpha Centauri, would be three times as far away as the *real* Moon is from the *real* Earth!

Worked example 2.15

The plan of a conference room is drawn to a scale of 1 : 200. If the room measures 45 mm by 70 mm on the plan, what are the real dimensions of the room? Explain your solution.

Solution

We need to find the real dimensions of the room.

Every 1 unit on the drawing represents 200 units in the conference room.

45 mm is $45 \times 200 = 9000$ mm or 9 m

70 mm is $70 \times 200 = 14\,000$ mm or 14 m

So, the dimensions of the room are 9 m by 14 m

Looking back, if the real size of the room is 9 m by 14 m, then a 1 : 200 scale would make each real measurement 200 times smaller.

Therefore, drawing sizes will be:

$\frac{9\text{ m}}{200} = \frac{9000\text{ mm}}{200} = 45\text{ mm}$ and $\frac{14\text{ m}}{200} = \frac{14\,000\text{ mm}}{200} = 70\text{ mm}$,

which are the measurements given in the question.

Worked example 2.16

The diving tank at the Neutral Buoyancy Laboratory in Houston, Texas, is 62 m long, 31 m wide and 12.34 m deep.

If NASA wanted to build another diving tank with the same ratio as the Neutral Buoyancy Laboratory tank, but 15 m deep, what would be the dimensions of the new diving tank?

Solution

The ratio of the dimensions of the Neutral Buoyancy Laboratory tank is

62 : 31 : 12.34

If this ratio is going to be kept with the 15 m deep tank, then the equivalent ratio will be

$62 : 31 : 12.34 = x$ (length) : y (width) : 15 (depth)

Thus, we can solve the following two proportions to find the new length and width:

$\frac{62}{x} = \frac{12.34}{15}$ and $\frac{31}{y} = \frac{12.34}{15}$

Therefore, $x = \frac{62 \times 15}{12.34} = 75.36\text{ m}$ and $y = \frac{31 \times 15}{12.34} = 37.68\text{ m}$

Practice questions 2.2.2

1 Find the missing variable in each of these proportions.

a $\frac{8}{10} = \frac{24}{x}$ b $\frac{15}{27} = \frac{y}{81}$ c $\frac{w}{25} = \frac{88}{100}$ d $\frac{11}{z} = \frac{66}{162}$

Hint Q2

For example, 8 : 5 = 1.6 : 1

2 Express each of these ratios in the form of $a : 1$

a 75 : 15 b 44 : 20 c 16 : 6 d 144 : 18

3 Find the missing variable in each of these proportions.

a $\frac{10}{25} = \frac{70}{x+2}$ b $\frac{11}{y-3} = \frac{33}{63}$ c $\frac{2.1}{3.6} = \frac{1-z}{12}$ d $\frac{7w}{14} = \frac{20}{36}$

4 The northern hemisphere accounts for about 90% of the world's total population. If there are currently about 7 820 000 000 people, find:

a the population of the northern hemisphere

b the population of the southern hemisphere.

5 If Ayubke and Olga share €420 in the ratio of 3 : 4, how much do they each receive?

6 The scale of a drawing is 1 : 300. Convert each of these drawing distances into the real distances.

a 3 cm b 1.2 cm c 2 m d 250 mm

7 Determine if each of these relationships is proportional.

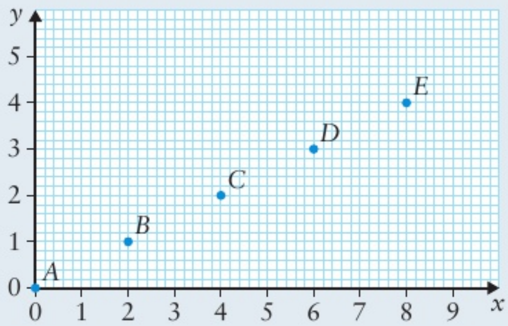
a

x	y
1	4
3	6
5	8
9	12
15	18

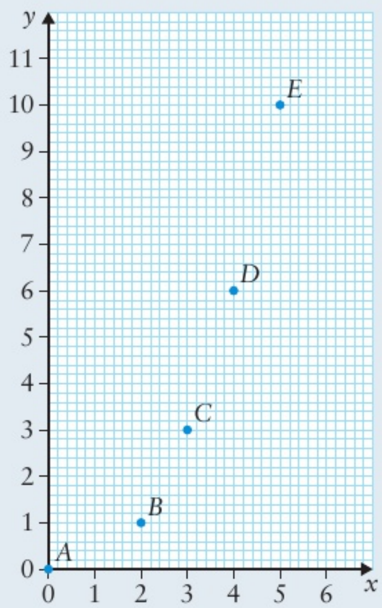
b

x	y
1	10
2	14
3	18
4	22
5	26

c



d



8 The dimensions of the model of a fuel reservoir are in the ratio 4 : 5 : 6. That is length : width : height. If the height of the reservoir is 24 metres, find the length and width.

2.2.3 Rates

Explore 2.8

Can you answer the following questions?

a A car travels with an average speed of 100 km/h. How long will it take to travel 650 km? Justify your answer.

b If light travels at 300 000 km/s, how far does it travel in an hour?

A **rate** is a comparison of two unlike quantities. The most common rate you will come across is **speed**, which is the ratio of distance travelled to time taken. Rates are usually written in **the simplest form** or in **the unit rates**, where the units of the rates are always shown. For example: 500 litres for 10 bottles is shown as 50 litres per 1 bottle or 50 L/bottle.

Worked example 2.17

Write each of these expressions as rate in its simplest form.

a A student walks 5 km in 0.5 hours **b** 20 ice creams cost €40

Solution

a $\frac{5\text{ km}}{0.5\text{ hour}} = 10\text{ km/h}$

b $\frac{20\text{ ice creams}}{€40} = 0.5\text{ ice creams / €}$

or $\frac{€40}{20\text{ ice creams}} = €2\text{ / ice cream}$

Worked example 2.18

Kenyan athlete Eliud Kipchoge won the 2019 London marathon with the time of 2 hours, 2 minutes and 37 seconds. He ran 42.195 km in the marathon. What was his average speed?

Eliud Kipchoge

Solution

Understand the problem

We need to find Kipchoge's average speed, using the distance and his running time.

Make a plan

Speed is the ratio of distance travelled to time taken. Thus, find the rate in its simplest form.

Carry out the plan

$$\frac{42.195\,\text{km}}{\left(2+\frac{2}{60}+\frac{37}{3600}\right)\text{hours}} = 20.65\,\text{km/h}$$

Look back

If we multiply the speed by the time travelled, it should give the total distance travelled. Therefore, $20.65\,\text{km/h} \times 2.04\text{ h} \approx 42.126\,\text{km}$, which confirms the solution. The difference is due to rounding!

Worked example 2.19

Francesco can paint a $30\,\text{m}^2$ wall in 4 hours. Find his speed in:

a m^2/h **b** cm^2/s.

Solution

Understand the problem

We need to find Francesco's speed for painting at two different unit rates.

Make a plan

a Find the unit rate as m^2/h

b Convert the unit rate into cm^2/s

Carry out the plan

a $\frac{30\,\text{m}^2}{4\,\text{h}} = \frac{7.5\,\text{m}^2}{1\,\text{h}} = 7.5\,\text{m}^2/\text{h}$

b $\frac{7.5\,\text{m}^2}{1\,\text{h}} = \frac{75\,000\,\text{cm}^2}{3600\text{ s}} = 20.8\dot{3}\,\text{cm}^2/\text{s}$

Look back

a If Francesco can paint at $7.5\,\text{m}^2/\text{h}$, then he can paint $7.5 \times 4 = 30\,\text{m}^2$ in 4 hours, which is correct.

b If Francesco can paint at $20.8\dot{3}\,\text{cm}^2/\text{s}$, then he can paint

$$\frac{20.8\dot{3}}{1} \times \frac{\frac{1}{10\,000}}{\frac{1}{3600}} \times \frac{\text{m}^2}{\text{h}} = \frac{20.8\dot{3}}{1} \times 0.36\,\text{m}^2/\text{h} = 7.5\,\text{m}^2/\text{h}$$

Practice questions 2.2.3

1 Simplify each of these rates and calculate the unit rates.

a 250 km in 2 hours
b 200 kg for 8 m^2
c 500 litres for 4 minutes
d 50.8 krone for 4 litres

2 A mine can supply 3000 tonnes of iron per day. Find how many tonnes the mine can supply in:

a 6 weeks
b one year.

3 Peter can walk at 4 km/h. How far can he walk in 5 hours at the same speed?

4 How long will it take to travel 1200 km at 90 km/h?

5 If Elif saves €25 per week, how long will it take her to save €1000?

6 Usain Bolt, sometimes called the fastest man on Earth, holds the 100 metres and 200 metres world records with 9.58 seconds and 19.19 seconds respectively. Which of his average speeds (100 m or 200 m) is faster?

Challenge Q7

7 Henri's car uses 25 litres of petrol on a journey of 300 km, and Carole's car uses 48 litres of petrol on a journey of 580 km. Which car is more economical? Why?

Challenge Q8

8 Light travels at a speed of 300 000 km/s.

a How far does it travel in 5 days?
b A star is 12 000 000 000 000 000 000 km away from Earth. How long does it take for the light from this star to reach us on Earth?

Self assessment

- I can change a percentage into a decimal.
- I can convert a decimal to a percentage.
- I can write a percentage as a fraction or a decimal.
- I know how to find a percentage of a number.
- I can calculate the percentage of profit or loss.
- I can represent two quantities as a percentage.
- I know what a ratio is.
- I can simplify ratios.
- I can split a number into a ratio.
- I know that two equivalent ratios are in proportion.
- I can find the constant of a proportion.
- I can find a missing term of a proportion.
- I know that if a relationship is proportional then it is a linear relationship.
- I can draw the graph of a linear relationship.
- I know that graphs of linear relations are straight lines.
- I know what a rate is.
- I can write a rate in its simplest form.
- I can convert different units.
- I can find real lengths from measurements in a scale drawing.

Check your knowledge questions

1 Write down these ratios in their simplest form.

a $20:12$ **b** $1.2:3.4$ **c** $\frac{210}{330}$ **d** 66%

2 Calculate the missing value in each of these proportions.

a $8:7 = 12:x$ **b** $44:66 = y:36$

c $\frac{z}{15} = \frac{0.05}{0.75}$ **d** $\frac{12}{w} = \frac{36}{51}$

3 The ratio of cats to dogs is $4:6$

a Calculate the number of dogs, if there are 16 cats.

b What percentage of the animals are cats, in the two pictures?

Challenge Q3b

4 The four Hawaiian islands with the largest populations, as of 2018, are given in the table.

Island	Population
Oahu	980 080
Hawaii	200 983
Maui	153 997
Kauai	72 133

Find the population ratios of:

a Hawaii to Oahu **b** Maui : Hawaii

c Kauai : Oahu **d** Hawaii : Total of the four islands

Challenge Q4d

5 Calculate each of these rates in its simplest form.

a 60 km travelled in 3 hours **b** 12 kg for every 6 m^2

c 45 grams per every 15 litres **d** 15 boys to 20 girls

6 A drawing scale is given as 1 : 300. Calculate the length on the drawing of each of these real distances.

a 12 m **b** 5 km **c** 70 m **d** 3000 cm

Challenge Q7

7 Calculate the missing values in these ratios.

$12 : x : 30 = 36 : 45 : y$

Challenge Q8

8 Thomas says that a triangle is an isosceles right-angled triangle if the interior angles of the triangle are in the ratio of 1 : 1 : 2. State whether Thomas is correct. Justify your answer.

Challenge Q9

9 Are the given relationships proportional? State the constant of proportion if the relationship is proportional.

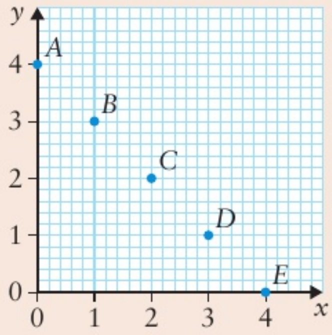
a

x	y
1	5
2	8
3	11
4	14
5	17

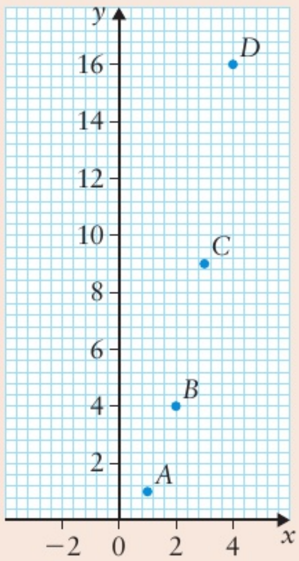
b

x	y
0	0
3	1
6	2
9	3

c



d



10 Jessica and Sebastian will travel from Hanover to Amsterdam for the weekend.

When they enter their destination into their satnav system, two route options are given:

Route 1: 433 km and 4 hours and 12 minutes

Route 2: 375 km and 3 hours and 50 minutes

a Find the average speed for each route.

b Which route should they take? Why?

11 The table overleaf lists the top 10 countries by their wind power capacity in 2019.

Calculate the ratio of the wind power capacity of:

a China to Spain

b Canada to India

c Germany to United States

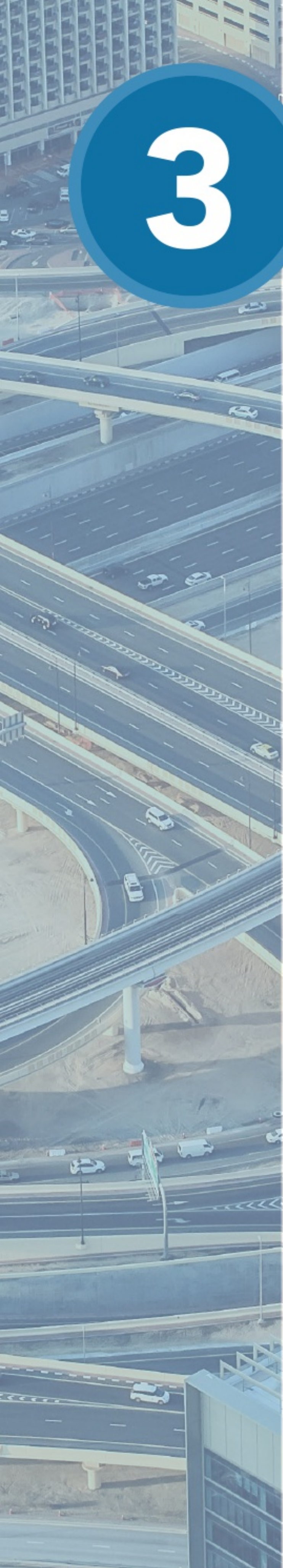
d Italy to Brazil.

Country	Wind capacity (Megawatts)
China	236 402
United States	105 466
Germany	61 357
India	37 506
Spain	25 808
United Kingdom	23 515
France	16 643
Brazil	15 452
Canada	13 413
Italy	10 512

12 A scale drawing is in the ratio 1 : 1000. Calculate the lengths on the drawing for each of these real distances.

a 6 m b 12 m c 120 cm d 2500 mm

3 Algebraic expressions



3 Algebraic expressions

KEY CONCEPT

Form

RELATED CONCEPTS

Generalisation, Patterns, Simplification, Systems

GLOBAL CONTEXT

Globalisation and sustainability

Statement of inquiry

By understanding how we can generalise patterns in different forms and simplify processes, we can improve systems of sustainability.

Factual

- Are the numerical methods of long addition and multiplication transferable to addition and multiplication of algebraic expressions?
- Can we divide one algebraic expression by another with a method similar to chunking or long division? Could geometry help us divide algebraic expressions?

Conceptual

- What does simplification mean?

Debatable

- Is it always possible to simplify an algebraic expression?

Do you recall?

1 Evaluate the following expressions.

a $-3 + 4$

b $-3 - (-4)$

c $-3 - 4$

2 Evaluate the following expressions.

a 3×4

b $12 \div (-3)$

c $-12 \times (-3)$

3 Find the HCF (Highest Common Factor) of these sets of numbers.

a 48 and 60

b 42 and 70

c 60, 90 and 132

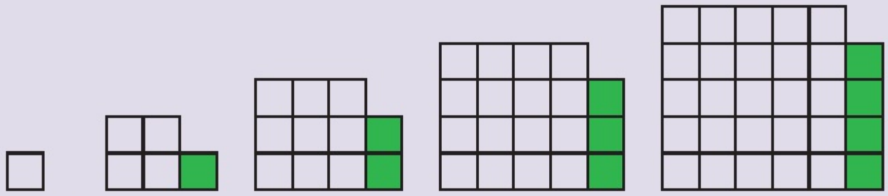
4 Find the prime factorisation of these numbers.

a 48 **b** 60 **c** 98 **d** 315

3.1 Simplifying algebraic expressions

3.1.1 Recognising expressions

Explore 3.1



n Figure 1 Figure 2 Figure 3 Figure 4 Figure 5

This is a sequence of five figures. Each figure has a certain number of unshaded squares and green squares. For example, Figure 3 is made up of 9 unshaded and 2 green squares.

Can you draw Figure 6? Can you draw Figure 10? What about figure n? That is, can you describe the pattern both in words and using n, the number corresponding to any figure?

An algebraic expression is an expression containing constants, variables, and any algebraic operations that connect them.

For example, $5n$, $y - 8$, $\frac{3z}{11}$, $3x^2 - 5xy + 7$ are algebraic expressions.

$5x^2$ is an expression, and so is $5x^2 + 12x$ and $5x^2 + 12x + 7$

The difference is that $5x^2$ has just one **term** and it is called a **monomial**.

$5x^2 + 12x$ has two terms and it is called a **binomial**.

$5x^2 + 12x + 7$ is an expression with three terms and is called a **trinomial**.

If there are more than three terms, we simply call it a **polynomial**.

Notice that the first two terms have letters. These are called **variable terms**. However, the third term, 7, does not have a variable with it, so this is called a **constant term**.

Here are two diagrams that may help you remember all the names we use.

Coefficient Constant

$$\underbrace{5x^2}_{\text{term}} + \underbrace{12x}_{\text{term}} + \underbrace{7}_{\text{term}}$$

Monomial	$5x^2y$
Binomial	$5x^2y + 13xy^3$
Trinomial	$5x^2y + 13xy^3 - 0.75x^2$

An equation is a mathematical phrase where two sides of the phrase are connected by an equals sign (=). For example, $5x + 3 = 33$ is an algebraic equation where 33 is the right-hand side (RHS) and $5x + 3$ is the left-hand side (LHS) of the equation.

3.1.2 Addition and subtraction

Simplifying an expression is the process of combining terms by carrying out operations such as removing grouping symbols, addition, and subtraction.

Explore 3.2

Can you simplify both expressions?

$8a^2 - 3a^2 + 2a^2$ and $8a^3 - 3a^2 + 2a + 1$

Justify your answer.

Like (similar) terms are terms that have the same variables and powers. The coefficients do not need to match. For example, in $5x + 2x^3 - 11 + 7x - 3x^2 + x^3$, $5x$ and $7x$ are like terms and $2x^3$ and x^3 are also like terms. Notice that the coefficients do not match.

Worked example 3.1

Simplify these expressions.

a $4y - 3x + 7 - y - 11 + z + 8x$ **b** $x^2 + 5x - 6 + 3x - 4$

Solution

a **Understand the problem**

We need to see if we can write an equivalent expression with fewer terms.

Make a plan

Combine like terms by adding/subtracting their coefficients.

Carry out the plan

Here, all the xs are in blue, ys are in red, zs are in black, and constants are in green: $4y - 3x + 7 - y - 11 + z + 8x$

Rearranging gives us $-3x + 8x - y + 4y + z + 7 - 11$, which you may prefer to put in brackets:

$(-3x + 8x) + (-y + 4y) + z + (7 - 11)$

Now we can combine the terms in each set of brackets.

This gives $5x + 3y + z + (-4) = 5x + 3y + z - 4$

Look back

To check our answer we could choose values for x, y and z and substitute them into the original expression and the final expression and see if the values match.

b The key here is to realise that, even though x^2 and $5x$ both have x's in them, they are not 'like' terms.

We can rearrange and put the like terms in brackets:

$x^2 + 5x - 6 + 3x - 4 = x^2 + (5x + 3x) + (-6 - 4)$

Combining the like terms gives $x^2 + 8x + (-10) = x^2 + 8x - 10$

Practice questions 3.1.2

1 Simplify these expressions.

a $x + x + x + x + y + y + 1 + 1 + 1$ **b** $2p + 7q + 4p + q$

c $3a + 4b + 5c + 6b + 2$ **d** $5x + 3 + 4x + 8$

e $9x + 5 - 3x - 1$ **f** $8x - 4 - 7x + 9$

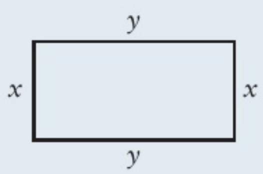
g $-3x + 5 - 2 + 4x$ **h** $3a - 2b - 6c - 4b - 8a + 5c$

i $2 + 3x^2 - 8x - x + 7 + 2x^2$ **j** $(7 - x) + (5x - 3)$

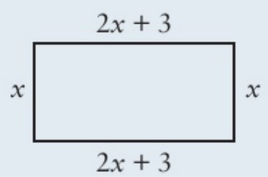
k $(2x - 1) - (x + 4)$ **l** $(7a + 2b - 4c) - (c - a - 5)$

2 Find the perimeter of each shape in terms of x and y.

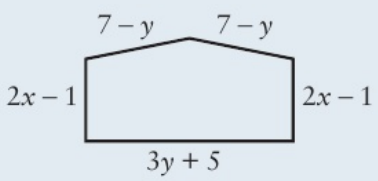
a



b



c



3.1.3 Multiplication and division

Explore 3.3

How would you write these expressions in their simplest form?

a $2 \times a$ **b** $14b \div 7$ **c** $c \times c$ **d** $5d \div d$

Worked example 3.2

Simplify each of these expressions.

a $9m \times 8n$ **b** $7y \times 3y$ **c** $5pq \times 8qr$

Solution

a $9m \times 8n = 9 \times 8 \times m \times n = 72 \times mn = 72mn$

b $7y \times 3y = 7 \times 3 \times y \times y = 21y^2$

c $5pq \times 8qr = 5 \times 8 \times p \times q \times q \times r = 40pq^2r$

In each case, we multiply the coefficients and combine the variables when possible.

Practice questions 3.1.3

1 Simplify:

a $5 \times z$

b $x \times x$

c $y \times y \times y$

d $z^2 \times z^3$

e $7 \times 3a$

f $4b \times 6b$

g $x \times y \times z$

h $2a \times 3b \times 4c$

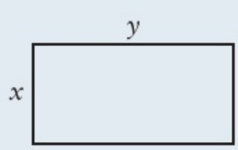
i $9x^5 \times 7x^8$

j $3q^2 \times 4p^5 \times q^4 \times \frac{1}{2}r^3$

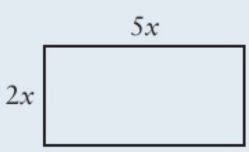
2 Find the area of each shape in terms of x, or x and y.

a

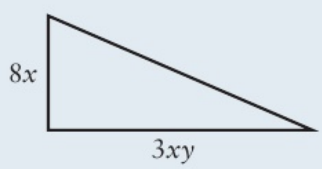
b



c



d



3.1.4 Simplifying division and algebraic fractions

Explore 3.4

How would you write these expressions in their simplest form?

a $15xy \div 3x$ **b** $12ac \div 8ab$ **c** $-6x^2 \div 18xy$ **d** $\frac{3b}{5} + \frac{2b}{4}$

Worked example 3.3

Simplify each of these expressions.

a $48mn \div 16m$

b $45xy \div 20yz$

c $-28abc^2 \div (-7ac)$

d $\frac{7x}{10} + \frac{3x}{4}$

Solution

a $48mn \div 16m = \frac{{}^{3}\cancel{48}\cancel{m}n}{{}_{1}\cancel{16}\cancel{m}} = 3n$

Express the division as a fraction and reduce it to its lowest terms.
Divide the numbers by their GCD, which is 16 in this case.
Looking at the variables, the *m*s cancel each other.

b $45xy \div 20yz = \frac{{}^{9}\cancel{45}x\cancel{y}}{{}_{4}\cancel{20}\cancel{y}z} = \frac{9x}{4z}$

Divide the numbers by their GCD, which is 5 in this case.
The *y*s cancel each other.

c $-28abc^2 \div (-7ac) = \frac{{}^{4}\cancel{-28}\cancel{a}bc^{\cancel{2}1}}{{}_{1}\cancel{-7}\cancel{a}\cancel{c}} = 4bc$

Divide the numbers by their GCD, 7.
The *a*s cancel each other, as does one *c* in the numerator with the *c* in the denominator.

Note that $\frac{-28}{-7} = +4$

d $\frac{7x}{10} + \frac{3x}{4} = \frac{7x}{10} \times \frac{2}{2} + \frac{3x}{4} \times \frac{5}{5} = \frac{14x}{20} + \frac{15x}{20} = \frac{29x}{20}$

We look for the LCM of 10 and 4, which is 20.

Multiply the first term by $\frac{2}{2}$ and the second by $\frac{5}{5}$ to make both denominators 20. Add the numerators of the resulting fractions.

Reminder a

Reminder: GCD is the greatest common divisor, also known as HCF, the highest common factor.

Reminder d

LCD is the lowest common denominator, also known as LCM, the lowest common multiple.

Practice questions 3.1.4

1 Simplify:

a $\frac{p^5}{p}$ b $\frac{b^7}{b^3}$ c $\frac{3t^{10}}{t^4}$ d $\frac{z^9}{5z^2}$

e $\frac{32k^{15}}{8k^7}$ f $\frac{28x^6}{42x^9}$ g $\frac{28a^5b^8}{24ab}$ h $\frac{56pq^8}{7p^4q^2}$

2 Simplify:

a $24x^3 \div 6$ b $9y^5 \div y^2$

c $63z^8 \div 21z^6$ d $18a^4b^9 \div 10ab^2$

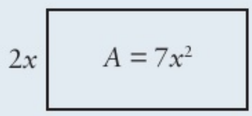
e $52p^8 \div 39p^7q^9$ f $37m^3n^2p^4 \div 74mn^2 \times 2mp$

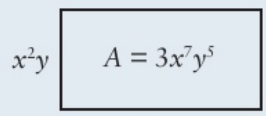
3 Simplify:

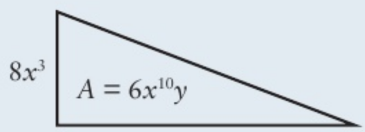
a $\frac{2x^5y^{13}}{15z^6} \times \frac{6xz}{7y^9}$ b $\frac{4x^5}{9y^2} \div \frac{10x^3y}{21z^8}$

c $\frac{8a^5}{9b^3c^6} \div \frac{15b^4}{16a^2c^3}$ d $84x^5y^4 \times \frac{55yz^9}{49x^2} \div \frac{60z^8}{91xy^5}$

4 Find the base of each shape in terms of x and/or y, given the height and area.

a


b


c


Hint Q3

Remember, to divide one fraction by another, we multiply the first by the reciprocal of the second.

$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$

3.2 Expanding brackets

3.2.1 Expanding single brackets

The school canteen is preparing for a large sports event. They want to offer 'Healthy Snack Sacks' to include a small bag of almonds, two bananas, and 5 squares of dark chocolate. They are expecting 150 people to attend. Find out how many almonds, bananas and squares of chocolate they need to purchase.

If we refer to one bag of almonds as a, one banana as b, and one square of chocolate as c, then one snack sack is $a + 2b + 5c$

> **Fact**
>
> To expand brackets, you multiply everything inside the brackets by the term outside the brackets.

If we have 150 sacks, then we have

$$150(a + 2b + 5c) = 150 \times a + 150 \times 2b + 150 \times 5c$$
$$= 150a + 300b + 750c$$

This means 150 bags of almonds, 300 bananas and 750 squares of chocolate.

If they manage to sell 140 bags of almonds, 245 bananas and 650 of the chocolate squares, how much are they left with?

$$150a + 300b + 750c - (140a + 245b + 650c)$$

This will require that we remove the bracket. Thus

> **Fact**
>
> Here we multiply each term inside the brackets by the negative sign, which is really a (-1). Multiplying a term by -1 changes its sign.

$$150a + 300b + 750c - 140a - 245b - 650c$$
$$= 150a - 140a + 300b - 245b + 750c - 650c$$
$$= 10a + 55b + 100c$$

Worked example 3.4

Expand the brackets in these expressions.

a $6(7x - 3)$ **b** $-2(2x - 1)$ **c** $\frac{5}{2}(6x + 10)$ **d** $x(3 + 4x)$

Solution

a The 6 outside the brackets means we have 6 lots of what is inside, so we have 6 lots of $7x$ and 6 lots of (-3).

We can change $6(7x - 3)$ to $6 \times 7x + 6 \times (-3)$ (multiplying each term inside the brackets by 6), which gives $42x + (-18) = 42x - 18$

b Just as we multiplied each term inside the brackets by 6 in part a, here we multiply each term by (-2).

$-2(2x - 1)$ means $-2 \times 2x + (-2) \times (-1) = -4x + 2$

c $\frac{5}{2}(6x + 10)$ means $\frac{5}{2} \times 6x + \frac{5}{2} \times 10 = 3x + 4$

d $x(3 + 4x)$ means $x \times 3 + x \times 4x = 3x + 4x^2$

Worked example 3.5

Two farmers are fencing in part of their land. They have purchased 300 metres of fencing and will create a rectangular area for their cows to graze on. A river runs straight along the edge of their land.

Write expressions for the amount of land they can fence in:

a if the grazing land does not border the river

b if the grazing land borders the river.

Solution

Understand the problem

We need to find the area of a rectangle, so we must know the length and width of the rectangle. The farmers can fence in an area that does or does not border the riverbank, so we consider two situations.

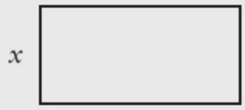
Make a plan

Let's draw two diagrams to help work out the base and height of the rectangles possible in each situation. Then we can multiply these quantities to find the area in square metres.

For the sake of each diagram, let x be the width of the rectangle.

Carry out the plan

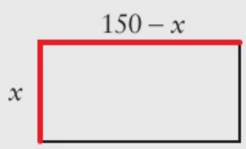
a First, let's draw the area *not* bordering the riverbank.



To find the length of the rectangle, remember that the length plus the width gives us half the perimeter.

The perimeter will be 300 m, since that is the total length of the fencing, so the length plus the width is half of 300 m, or 150 m.

x + length = 150 gives us length = $150 - x$



Hence, the area of the rectangle is $x(150 - x) = x \times 150 - x \times x$
$= (150x - x^2)\,\text{m}^2$

Look back

To check that we have not made a careless mistake in our algebra, we could choose a possible value for x and test it out in the formula to see if we get the correct answer.

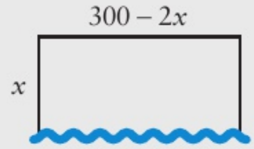
For example, if the length of the rectangle were 40 m, the height would have to be 110 m to give a perimeter of 300 m ($40 + 110 + 40 + 110 = 300$). The area would then be $40 \times 110 = 4400\,\text{m}^2$

Checking $x = 40$ in our result, $150x - x^2$, gives
$150 \times 40 - 40^2 = 6000 - 1600 = 4400\,\text{m}^2$

Carry out the plan

b Now consider the second situation, where the fenced area borders the river. So, there will fencing on only three sides of the rectangle.

In our diagram, the left and right sides of the rectangle are both x metres long, and subtracting these from the total length of 300 m of fencing gives $(300 - 2x)$ m remaining for the top base of the rectangle.



Hence, the area of the rectangle is $x(300 - 2x) = (300x - 2x^2)\,\text{m}^2$

Look back

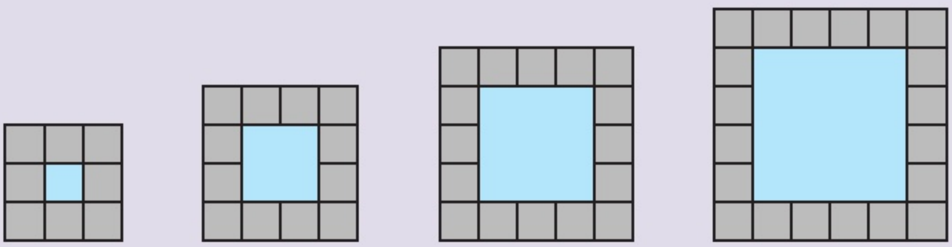
We can do the same check as we did for the first situation to test the accuracy of our algebraic manipulation.

Explore 3.5

Suppose for the second situation, the farmers chose x to be the length of the side of the rectangle opposite the riverbank. Can you find an expression for the area?

Explore 3.6

A path is made around a square pond using 1 m by 1 m square paving stones. The first few possible ponds are shown here:



Can you describe an efficient method for counting the number of paving stones in each diagram? Let us call this your Method 1. Can you think of another way to see it and describe this? This is your Method 2.

Use your method to work out how many paving stones would surround a square pond of length 100 metres.

How many paving stones would surround a square pond of length x metres? Use Method 1, and simplify your result. Do you get the same result when using Method 2?

Practice questions 3.2.1

1 Expand the brackets in each of these expressions.

a $6(x + 4)$ **b** $\frac{1}{2}(4 - 6x)$ **c** $\frac{2}{3}(6x + 9)$ **d** $\frac{6(8x - 10)}{4}$

2 Simplify these expressions.

a $9(x + 2) + 3(4x - 1)$ **b** $5(3 - 2x) + 4(-7x - 4)$

c $6(2x - 5) - 7(3 - x)$ **d** $-9(3x - 1) - (10 - 4x)$

3 Simplify these expressions.

a $\frac{12x - 8}{4} - 3(x - 5)$ **b** $\frac{3}{2}(14x - 8) - \frac{18x + 6}{3}$

c $12(4x - 3) + \frac{10(6x - 3)}{15}$

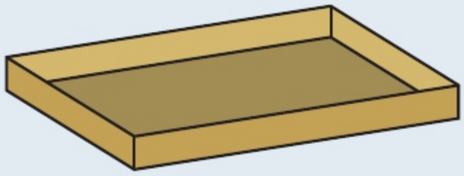
4 Write an expression for the perimeter of the following regular polygons.

a A regular pentagon, with each side measuring $4a + b$

b A regular octagon, with each side measuring $7c - 2d$

c A regular nonagon, with each side measuring $\frac{2}{3}e - \frac{1}{2}$

5 A company makes cardboard boxes using recycled paper. Their medium-sized box is made by cutting the corners off a 50 cm by 60 cm sheet of cardboard and then folding the flaps up to form the sides of the open-topped box.



If the corners that are cut out measure x cm by x cm, find:

a the width of the box, in terms of x

b the length of the box, in terms of x

c the area of each of the side flaps of the box, in terms of x.

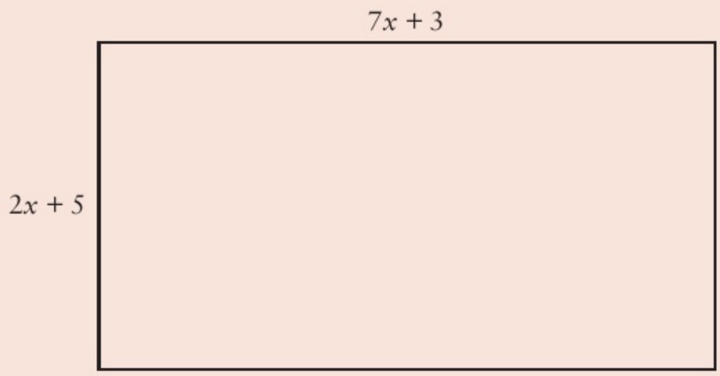
3.2.2 Expanding a pair of brackets

Investigation 3.1

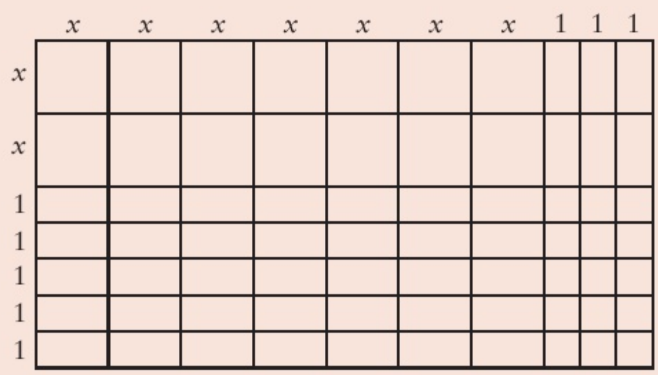
Part 1

To explain what $(2x + 5)(7x + 3)$ means, we can do the following:

Draw a rectangle to represent the multiplication $(2x + 5)(7x + 3)$.



This rectangle could be divided up into a grid of smaller rectangles.



How many different-sized rectangles are there? This is how many terms the simplification of $(2x + 5)(7x + 3)$ should have.

Hint

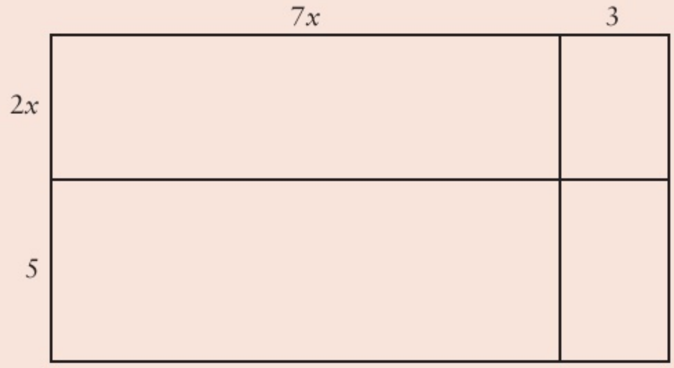
Remember:

- squares are also rectangles
- if two shapes are just reflections or rotations of one another, they are still **congruent** and considered the same size.

Part 2

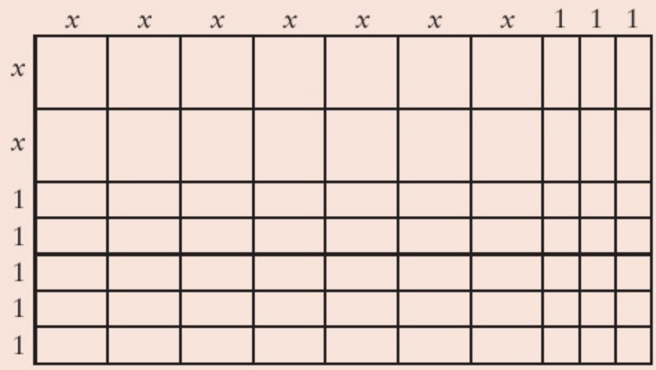
1 What are the sizes of the smaller rectangles in the diagram below? Hence, what is the total area of the large rectangle?

2 How could the diagram below have made it more efficient for us to get this result?



Part 3

1 Can you write simplified expressions for the length and width of this rectangle?



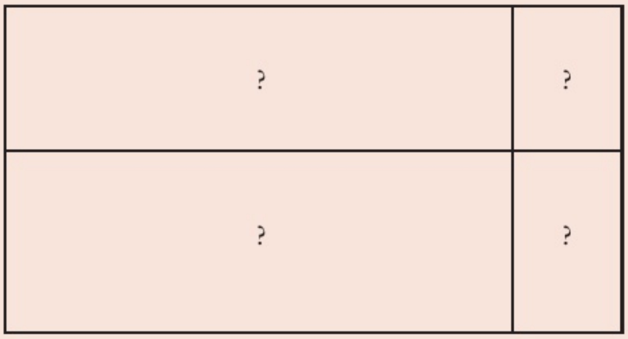
2 How could you use them to find the area of the rectangle?

3 Can you write an expression for the area of the rectangle?

Here is another version of the same rectangle.

4 Can you find the value of each question mark?

5 How about here?



Part 4

Summarise what you learned in this investigation.

Worked example 3.6

Expand the brackets in each of these expressions.
You may find it helpful to draw a diagram.

a $(x + 2)(x - 3)$ **b** $(3x - 1)(8x + 5)$

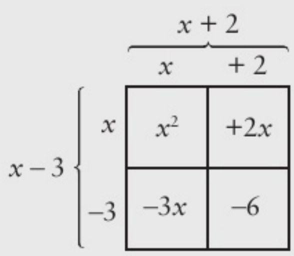
c $(x - 6)(7x - 1)$ **d** $(a + b)(c + d + e)$

Solution

a Multiply each term in the second set of brackets by each term in the first set of brackets:

$$\begin{aligned}(x + 2)(x - 3) &= x(x - 3) + 2(x - 3)\\ &= x^2 - 3x + 2x - 6\\ &= x^2 - x - 6\end{aligned}$$

This can be seen more clearly by using the 'box method' for multiplication:



b Multiply each term in the second set of brackets by each term in the first set of brackets:

$$\begin{aligned}(3x - 1)(8x + 5) &= 3x(8x + 5) - 1(8x + 5)\\ &= 24x^2 + 15x - 8x - 5\\ &= 24x^2 + 7x - 5\end{aligned}$$

c $(x-6)(7x-1) = x(7x-1) - 6(7x-1)$

$$= 7x^2 - x - 42x + 6$$

$$= 7x^2 - 43x + 6$$

d $(a+b)(c+d+e) = a(c+d+e) + b(c+d+e)$

$$= ac + ad + ae + bc + bd + be$$

This can be shown visually as:

$$\begin{array}{cc|c|c|c|}
 & & \multicolumn{3}{c}{\overbrace{\hspace{6em}}^{c+d+e}} \\
 & & c & d & e \\
\hline
\multirow{2}{*}{$a+b$} & a & ac & ad & ae \\
\cline{3-5}
 & b & bc & bd & be \\
\hline
\end{array}$$

Worked example 3.7

Simplify the expression $(x+a)(x-a)$.

Hence, simplify the expansions of $(x+7)(x-7), (x-5)(x+5)$ and $(20-3)(20+3)$.

Solution

$(x+a)(x-a) = x(x-a) + a(x-a) = x^2 - ax + ax - a^2$
The two x terms ($-ax$ and $+ax$) cancel each other out, leaving us with $x^2 - a^2$

The result is the square of the first number in the brackets minus the square of the second number in the brackets (disregarding its sign).

Hence, $(x+7)(x-7) = x^2 - 7^2 = x^2 - 49$ and
$(x-5)(x+5) = x^2 - 5^2 = x^2 - 25$

$(20-3)(20+3) = 17 \times 23$. Using the expression we found gives us a simpler way to calculate this: $(20-3)(20+3) = 20^2 - 3^2 = 400 - 9 = 391$

Reflect

How could you use the method of calculating 17×23 to work out 71×79?

Challenge

Challenge

Worked example 3.8

There are n people at a party, including the DJ. Everyone at the party shakes hands with all the other guests, except the DJ, who does not shake anyone's hand.

How many handshakes take place at the party? Simplify your answer.

Solution

We need to find the number of times the partygoers shake hands with one another.

We can multiply the number of guests at the party by the number of handshakes per person to get the total number of handshakes.

Since we don't count the DJ, there are $n - 1$ guests at the party. Each person shook hands with everyone at the party except themselves and the DJ, so each person completed $n - 2$ handshakes.

$$\begin{aligned}(n-1)(n-2) &= n(n-2) - 1(n-2)\\ &= n^2 - 2n - n + 2 = n^2 - 3n + 2 \text{ handshakes in total.}\end{aligned}$$

In solving problems, it is good practice to check each step before continuing.

If we test this out with four people, we get
$4^2 - 3 \times 4 + 2 = 16 - 12 + 2 = 6$ handshakes.

Let's say that the four people are the DJ, and guests A, B and C.

- Guest A will shake hands with B and C (2 handshakes).
- Guest B will shake hands with C and A, but we already counted the handshake between B and A (from A's perspective), so this is only one more.
- Guest C will shake hands with A and B, but we have already considered both of these, so the total should be just 3.

Thus, we must divide our result by 2, so we do not count each handshake twice.

$\frac{1}{2}(n^2 - 3n + 2) = \frac{1}{2} \times n^2 - \frac{1}{2} \times 3n + \frac{1}{2} \times 2 = \frac{1}{2}n^2 - \frac{3}{2}n + 1$ handshakes for n people.

Looking back, this would give a result of

$\frac{1}{2} \times 4^2 - \frac{1}{2} \times 3 \times 4 + \frac{1}{2} \times 2 = 8 - 6 + 1 = 3$ handshakes for four people, which is correct.

Practice questions 3.2.2

1 Draw rectangles with areas represented by each expression.

a $(x + 3)(x + 2)$ b $(x + 4)(x + 9)$

c $(3x + 4)(2x + 7)$ d $(5x + 3)(2x + 1)$

2 Simplify these expressions.

a $(x + 3)(x + 2)$ b $(x + 4)(x + 9)$

c $(x + 1)(x + 6)$ d $(x - 3)(x + 8)$

e $(x + 2)(x - 7)$ f $(x - 9)(x + 7)$

g $(x + 12)(x - 3)$ h $(x - 4)(x - 6)$

i $(x - 5)(x - 11)$ j $(x + 5)(x + 5)$

k $(x + 1)(x + 1)$ l $(x - 6)(x - 6)$

m $(x - 4)(x - 4)$ n $(x + 8)(x - 8)$

o $(x - 2)(x + 2)$ p $(x + 7)(x - 7)$

3 Simplify these expressions.

a $(x + 3)(5x + 2)$ b $(4x - 1)(x + 2)$

c $(2x + 6)(7x - 3)$ d $(2x + 3)(2x - 3)$

e $4(x - 7)(3x - 9)$ f $-3(2x + 1)(3x - 4)$

g $(x + 6)(2x^2 + 5)$ h $(x^3 + 2)(x^3 + 5)$

4 Simplify these expressions.

a $(x + 1)(x + 2)(x + 3)$ b $(x + 3)(x^2 + 5x + 7)$

c $3x(2x - 5)(4x + 3)$ d $(2x^2 - 5x + 3)(4x - 1)$

e $(x - 5)(x + 0)$ f $(3x - 0)(4x - 2)$

5 I am x years old. My brother is five years older than me, and my sister is three years younger than me.

a What is the product of our ages?

b What was the product of our ages five years ago?

c What will the product of our ages be five years from now?

6 A village has 100 people. x people in the village catch a cold.

a How many people in the village are still healthy, in terms of x?

The cold is spreading at a rate of y new people per day, where y is calculated by multiplying:

$0.05 \times$ number of sick people $\times$ number of healthy people.

b How many new people get sick per day, in terms of x? Simplify your answer.

7 The rectangular cross-section of a cuboid is 7 cm longer than it is wide.
Let the length of the cross-section be x cm.

a Find the area of the rectangular cross-section in terms of x.

b If the third dimension of the cuboid is 10 cm longer than the width of the cross-section, find the volume in cubic millimetres.

8 Consider the cardboard box from question 5 in Practice questions 3.2.1, made by cutting the corners off a 50 cm by 60 cm sheet of cardboard to form an open-topped box.

If the corners which are cut out measure x cm by x cm, find:

a the area of the bottom of the box, in terms of x

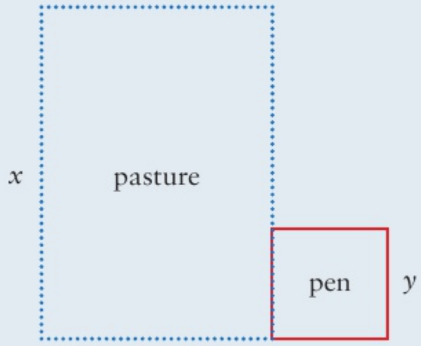
b the volume of the box, in terms of x

c the total surface area of the outside of the box, in terms of x.

9 An organic chicken farm is required by EU regulations to have at least $4\,m^2$ of pasture per chicken, and a maximum of six chickens per square metre may be kept in the pen where they shelter. A farmer has a square pen measuring y metres by y metres.

a Write an expression for the maximum number of chickens the farmer can have, in terms of y.

The farmer has 450 m of fencing to enclose a pasture for the chickens, which will be bordered by the pen. (The fencing is shown by a dotted line.)



If north is at the top of the diagram, then x m of fencing will be used on the west side of the pasture.

b How much fencing is needed for the east side of the pasture, in terms of x and y?

c Hence, how much fencing is needed for the south side of the pasture, in terms of x and y?

d What is the area of the pasture, in terms of x and y?

e Hence, what is the maximum number of chickens that can be on the pasture, in terms of x and y?

Reflect

Look carefully at the expressions in question 1 from Practice questions 3.2.2 and their simplified solutions.

How could you have gone straight from the question to the simplified answer with just some simple mental arithmetic, without writing down a working step on your paper?

Can you think of a rule that works in all cases? Do you notice any exercises that have interesting forms, either in the question or the answer? Can you come up with some rules that are true in special cases and describe these in words or with a formula?

3.3 Factorising

Factorising can be thought of as the opposite of expanding.

When you factorise an expression, you attempt to find the highest common factor (HCF) of all the terms in the expression.

Worked example 3.9

a Find the HCF of $4x^2y$ and $2xy^2$ **b** Factorise $4x^2y + 2xy^2$

Solution

a $4x^2y = 2xy \times 2x$ and $2xy^2 = 2xy \times y$

Hence, $2xy$ is the highest common factor of both terms.

b $4x^2y + 2xy^2 = 2xy \times 2x + 2xy \times y = 2xy(2x + y)$

Worked example 3.10

Find the HCF of the terms in each expression, and hence factorise the expression.

a $3x + 12$ **b** $x^5 + 4x^3$ **c** $8x^2 - 12x$

Solution

a The two terms here are $3x$ and 12. Since 12 is a constant, we look for the HCF of the numbers, or coefficients. The HCF of 3 and 12 is 3, giving us $3 \times x + 3 \times 4 = 3(x + 4)$

b The HCF of the coefficients of 1 and 4 is 1. The HCF of the variables x^5 and x^3 is x^3, so the HCF of the two terms of the expression is $1x^3 = x^3$ Hence, we can factorise: $x^5 + 4x^3 = x^3 \times x^2 + x^3 \times 4 = x^3(x^2 + 4)$

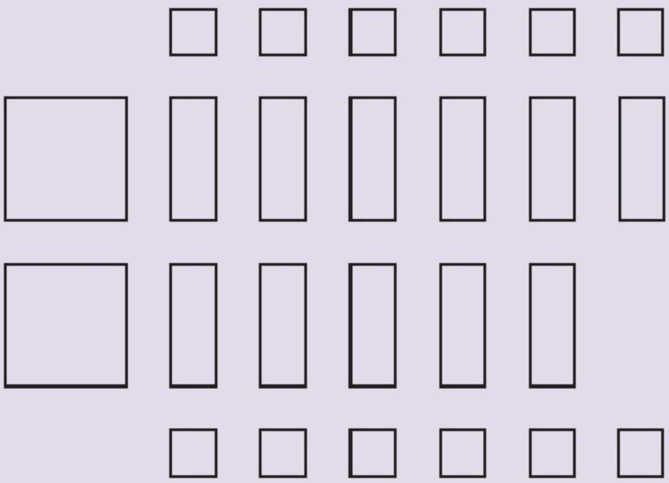
c In $8x^2 - 12x$, we can see the coefficients of 8 and -12 have 4 (or -4) in common, and each term has at least an x, so the HCF is $4x$, giving the factorisation $8x^2 - 12x = 4x \times 2x - 4x \times 3 = 4x(2x - 3)$

In Section 3.2.2, you were carrying out algebraic expansions on pairs of brackets to obtain simple quadratic expressions.

Given only a quadratic expression, how can you work out what was inside the brackets?

Explore 3.7

Form a single rectangle using all of the following puzzle pieces (small squares, large squares and oblong rectangles):



Is more than one solution possible? What algebraic expansion does this represent?

Hint

You could use link cubes or

If we think of the above rectangles as having dimensions x units by 1 unit, then the small squares have an area of 1 square unit, and the large squares have an area of x^2 square units. Then the total area of the shapes is $2x^2 + 11x + 12$, and the task of fitting the puzzle pieces into a rectangle is equivalent to factorising the expression.

Let's try this algebraically with some simpler examples. But first, a game.

Explore 3.8

Let's play a game, and we will try two versions of the game.

Version 1

I am thinking of two integers.

The sum of the integers is 21. What are my integers?

Well, lots of pairs are possible. In fact, the list in infinite.

0 and 21, 1 and 20, 2 and 19, 3 and 18, etc. are not the only pairs, because we also have negative numbers: -1 and 22, -2 and 23, and so on.

Another clue is that the product of my integers is 104. Can you work out my two integers now?

Version 2

I am thinking of two integers.

The product of the integers is 104. What are my integers?

This time there is not an infinite number of possibilities. The prime factorisation of 104 is $2^3 \times 13$, which gives us 2 and 52, 4 and 26, 8 and 13 and, of course, 1 and 104.

104 is positive, so it could also be the product of two negatives, which doubles our possibilities (-1 and -104, -2 and -52, -4 and -26, or -8 and -13), but still this is just eight possibilities.

We still need another piece of information to determine which is the correct pair, which is that the sum of my integers is 21. The negatives are all ruled out, so there are only four possibilities to test, which quickly leads to the solution of 8 and 13.

Which version of the game is easier? Keep this in mind as you approach the following problems.

Find the pair of integers with the following sums and products.

a sum = 17, product = 60

b sum = 17, product = 72

c sum = 18, product = 72

d sum = 1, product = -72

e sum = -6, product = -40

f sum = -27, product = 182

What reflections did you record about the expansions in question 1 of Section 3.2.2?

Hopefully you recognised that the questions were all of the form $(x + p)(x + q)$. That is, the **coefficient** of x in each bracket was 1, and p and q were positive or negative integers.

Looking at the results of each expansion shows that they were all of the form $x^2 + (p + q)x + pq$. That is, the coefficient of x^2 was always 1, the coefficient of x was the **sum** of the two constants, p and q, and the **constant** term was the **product** of p and q.

So how does our game help us to factorise quadratics where $a = 1$?

Fact

$ax^2 + bx + c$ is called a quadratic trinomial. It is **quadratic** since the highest power of x is 2. Each of the three terms has a constant coefficient. We will now look at quadratics where $a = 1$.

Worked example 3.11

Factorise these quadratics.

a $x^2 + 5x + 6$ **b** $x^2 + 5x - 6$

c $x^2 - 5x + 6$ **d** $x^2 - 5x - 6$

Solution

To factorise means to find the two binomials (in brackets) that expand to give these expressions. Since the coefficient of x^2 is 1 in each case, the first term of each binomial will be x. We just need to find the constant that is the second term of each binomial.

The last term of each quadratic is 6 or -6, so we need to think of which factors could give us a product of 6 or -6.

We can get a product of 6 in the following ways: 1×6, 2×3, $-1 \times (-6)$, $-2 \times (-3)$ and we can get -6 in these ways: -1×6, $1 \times (-6)$, -2×3, $2 \times (-3)$.

Now we need to find the factor pairs whose sums will give us the coefficient of x in each exercise, which is always 5 or -5.

For a product of 6, the possible sums are:

$1 + 6 = 7$, $2 + 3 = 5$, $-1 + (-6) = -7$, $-2 + (-3) = -5$

and the possible sums for a product of -6 are:

$-1 + 6 = 5$, $1 + (-6) = -5$, $-2 + 3 = 1$, $2 + (-3) = -1$

So, the pairs that work for each are:

a product $= 6$, sum $= 5 \Rightarrow 2$ and 3

b product $= -6$, sum $= 5 \Rightarrow -1$ and 6

c product $= 6$, sum $= -5 \Rightarrow -2$ and -3

d product $= -6$, sum $= -5 \Rightarrow 1$ and -6

To factorise means we should express these solutions as products of binomials in brackets, giving:

a $x^2 + 5x + 6 = (x + 2)(x + 3)$ **b** $x^2 + 5x - 6 = (x - 1)(x + 6)$

c $x^2 - 5x + 6 = (x - 2)(x - 3)$ **d** $x^2 - 5x - 6 = (x + 1)(x - 6)$

Worked example 3.12

Factorise $x^2 - 16$.

Solution

To factorise a quadratic with leading coefficient of 1, we look at the last term (constant) and the middle term (x term).

Since there is no visible x term, there are zero xs. Hence, we need to find two numbers whose product is (-16) and whose sum is 0.

Pairs of numbers that give 16 are 16 and 1, 8 and 2, and 4 and 4.

A negative product means the signs will be opposite, so we will have a positive number and a negative. The only way to get a sum of zero from two numbers is if the two numbers are the same with opposite signs.

So, the only pair that will work to give a product of (-16) is 4 and (-4)

Hence, the factorisation is $(x + 4)(x - 4)$.

To check, we can try expanding and seeing what it simplifies to.

$$\begin{aligned}(x + 4)(x - 4) &= x(x - 4) + 4(x - 4)\\ &= x^2 - 4x + 4x - 16\\ &= x^2 + 0x - 16\\ &= x^2 - 16\end{aligned}$$

Worked example 3.13

a Calculate $1\,000\,001^2 - 999\,999^2$ without a calculator.

b Find the prime factorisation of 8091.

Solution

a We could just square each number, but it would be tedious and prone to error. If we realise that the numbers being squared are close to one another, that makes things easier.

Let's refer to the smaller number as x. That is, $999\,999 = x$

Then $1\,000\,001 = x + 2$

So, $1\,000\,001^2 - 999\,999^2 = (x + 2)^2 - x^2 = x^2 + 4x + 4 - x^2$

The x^2 terms cancel each other out, so we are left with $4x + 4$, which will be even simpler to calculate if we factorise this.
$4(x + 1) = 4(999\,999 + 1) = 4\,000\,000$

b We could approach this by just trying to find factors. Clearly, 2 is not a factor, so if we want to check 3, we would sum the digits to see that $8 + 0 + 9 + 1 = 18$, which is divisible not only by 3 but also by 9, so 9 is a factor.

After this, however, we would unsuccessfully try dividing the number by 7, 11, 13… and realise this is potentially a very slow method.

If we notice that 8091 is close to 8100, which is a perfect square, then the task becomes quicker.

$8091 = 8100 - 9 = 90^2 - 3^2$, which is the difference of two square numbers.

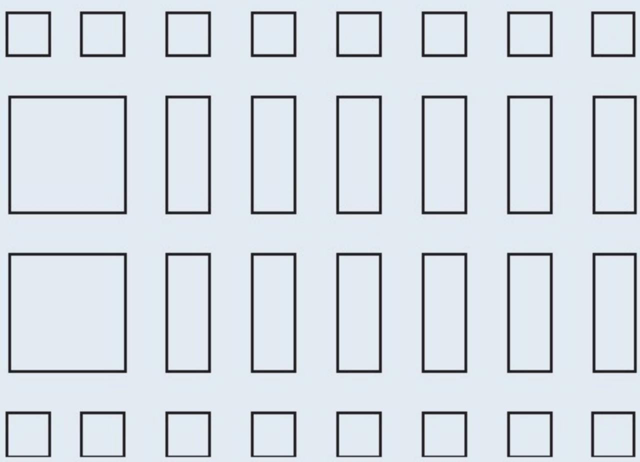
We found that we can factorise this as $(90 - 3)(90 + 3)$, which means that $8091 = 87 \times 93$

Now we just need to factorise 87 and 93, which are both multiples of 3.

$87 = 3 \times 29$ and $93 = 3 \times 31$. Hence, $8091 = 3^2 \times 29 \times 31$

Practice questions 3.3

1 Fit the puzzle pieces into a rectangle. Find a second solution that is not simply a reflection or rotation of the first.



2 Find the HCF, then factorise the expressions.

a $abc + abd + bcd$

b $12x + 18$

c $30x^2 - 20x$

d $16x^7y^{10} + 28x^9y^3$

e $4x^6 - 12x^5 + 6x^3$

3 Factorise these expressions.

a $x^2 + 8x + 7$
b $x^2 + 6x + 8$
c $x^2 + 8x + 15$
d $x^2 + 17x + 60$
e $x^2 - 3x + 2$
f $x^2 - 8x + 15$
g $x^2 - 17x + 42$
h $x^2 - 8x + 16$

4 Factorise these expressions.

a $x^2 - 6x - 40$
b $x^2 + 11x - 12$
c $x^2 - 6x - 16$
d $x^2 - 7x - 18$
e $x^2 + 2x - 24$
f $x^2 + x - 72$
g $x^2 + 0x - 9$
h $x^2 - 36$

5 Find the HCF, then factorise the expressions.

a $3x^2 - 18x + 15$
b $4x^2 + 28x + 24$
c $2x^2 + 16x + 30$

6 In Worked example 3.13, we could have chosen x to be 1 000 000 instead.

a Define 999 999 and 1 000 001 in terms of x.

b Use these definitions to solve the original problem.

c Use a similar method to calculate $\dfrac{2\,000\,002^2 - 1\,999\,998^2}{(1\,999\,998)(2\,000\,000)(2\,000\,002)}$

7 Write the prime factorisations of the following numbers.

a 851
b 4891
c 6319

Reflect

Why does question 1 in Practice questions 3.3 have two possible answers, while the first time you saw this type of puzzle, in Explore 3.7, it only had one answer?

Self assessment

- I can carry out operations using positive and negative numbers.
- I can identify like terms in algebraic expressions.
- I can add and subtract algebraic expressions.
- I can multiply and divide algebraic expressions by a constant.
- I can multiply powers of x.
- I can multiply a monomial and binomial.
- I can multiply two binomials.
- I can find common factors in algebraic expressions.
- I can factorise quadratics with a leading coefficient of 1.

Check your knowledge questions

1 $5x - 7 + ax + b$ simplifies to $11x + 3$. Find a and b.

2 Fill in the blanks in these equations to find the missing values.

a $3x + \square x = (\square + \square)x = 12x$

b $8x(\square x + \square) = \square \times \square + \square \times \square = 4x^2 + 24x$

c $(\square x - 2)(x + \square) = \square = 3x^2 + x + \square$

3 Draw a diagram to represent the multiplication $(x + 4)(3x + 2)$. Hence, write the product in simplified form.

4 Find the perimeter and area of a rectangle with dimensions:

a $5x$ cm by $(3x - 7)$ cm

i in cm and cm^2 ii in mm and mm^2.

b $(4x + 3)$ cm by $(9x - 8)$ cm

i in cm and cm^2 ii in mm and mm^2.

5 The HCF of $12x^py^3$ and qx^9y^r is $3x^7y^3$

a Which of the numbers p, q and r can you determine with certainty?

b Which of the numbers p, q and r must be odd?

c Which of the numbers must be greater than or equal to 3?

6 The number 37 can be expressed as $30 + 7 = 10 \times 3 + 7$
Similarly the two-digit number $AB = 10A + B$

a What is the square of the number AB? (That is, multiply AB by itself and simplify.)

b Show that, if the ones digit of AB is 5 (that is, if $B = 5$), this result can be written as $100A(A + 1) + 25$

c Use this result to mentally calculate 15^2, 25^2, 35^2, 45^2, 55^2, 65^2, 75^2, 85^2 and 95^2

7 A cyclist travels from A to B at a speed of $(5x - 2)$ km/h for x hours.

a Find the distance from A to B in terms of x.

b If she travels back from B to A cycling at 2 km/h slower than from A to B, how long does it take her?

c How many minutes longer is this?

8 A cube has length $4x$.

a Find the total edge length of the cube.

b Find the total surface area of the cube.

c Find the volume of the cube.

d Repeat steps **a** through **c** for a cube that is 1 cm longer.

9 a The lengths of two sides of an isosceles triangle are $6x - 9$ and $11x + 2$.

i Which is the shorter of the lengths?

ii What are the possible perimeters of the triangle?

b Two of the dimensions of a cuboid, which has a square cross-section, are $2x - 3$ and $7x + 4$. What are the possible volumes of the cuboid?

10 Factorise the following.

a $x^2 + 7x + 10$ b $x^2 - 5x - 84$

c $x^2 + 3x - 54$ d $x^2 - 13x + 36$

e $3 + 2x - x^2$ f $x^2 - 14x + 49$

g $x^2 - 121$ h $x^2 + 12x + 36$

i $9x^2 - 4$ j $9x^2 - 36$

k $3x^2 - 15x + 18$ l $4x^2 + 32x + 48$

Challenge Q11

Hint Q11

There are three possibilities.

Challenge Q12

11 A rectangle has area $4x^2 + 16x + 12$, where x is an integer. If the lengths of its base and height are also integers, find the possible dimensions of the rectangle.

12 Write the prime factorisations of the following numbers.

a 14 351 b 39 951

4 Pythagoras' theorem

4 Pythagoras' theorem

KEY CONCEPT

Logic

RELATED CONCEPTS

Patterns, Space, Validity

GLOBAL CONTEXT

Orientation in space and time

Statement of inquiry

Logic is an effective means for assessing the validity of what we discover through measurement and observation of objects in space.

Factual

- What is the sum of the two acute angles of a right-angled triangle?
- What is a Pythagorean triplet?

Conceptual

- How do we use the inverse of Pythagoras' theorem to show that an angle is a right angle?
- How do we use Pythagoras' theorem to measure irregular shapes?

Debatable

- Are visual 'proofs' of Pythagoras' theorem valid?

Do you recall?

1 Work out these calculations without a calculator.

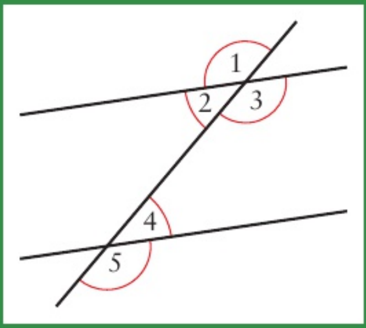
8^2 40^2 1.2^2 35^2

2 Work out these calculations without a calculator.

$\sqrt{49}$ $\sqrt{121}$ $\sqrt{2500}$ $\sqrt{1.44}$

3 Find the area of a square with side 8.62 cm.

4 The diagram shows two parallel lines intersected by a transversal. Copy and complete the lines below them.



Angles ______ and ______ are alternate.

Angles ______ and ______ are corresponding.

Angles ______ and ______ are vertical (or vertically opposite).

Angles ______ and ______ are co-interior.

Angles ______ and ______ are supplementary.

5 Expand these expressions.

$(x + 3)^2$ $(x + y)^2$ $(x - y)^2$ $(3x + 5y)^2$

4 Pythagoras' theorem

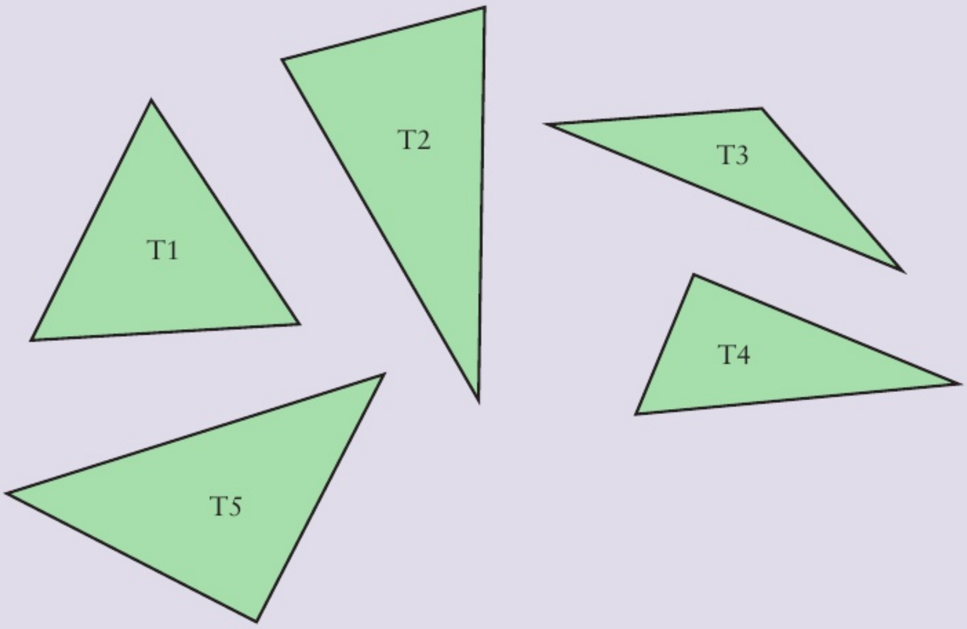
4.1 Right-angled triangles

Connections

Triangle properties

Explore 4.1

Use the word bank to write as many names as you can for each triangle.



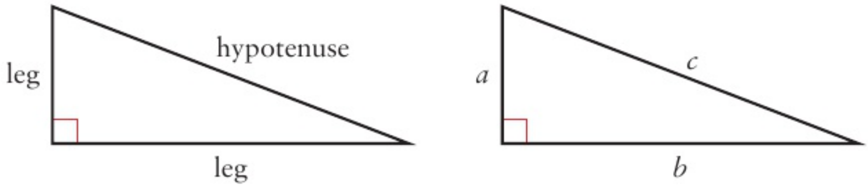
isosceles	scalene	obtuse	equilateral	right-angled	acute

Explore 4.2

What kind of angles does a right-angled triangle have?

Can a triangle have two right angles? Can a right-angled triangle have an obtuse angle?

The sides of a right-angled triangle adjacent to the right angle are called **legs**, and the side opposite the right angle is called the **hypotenuse**. Which side will always be the longest, and why? Must there always be a shortest side? Which will be the shortest side, if there is one?

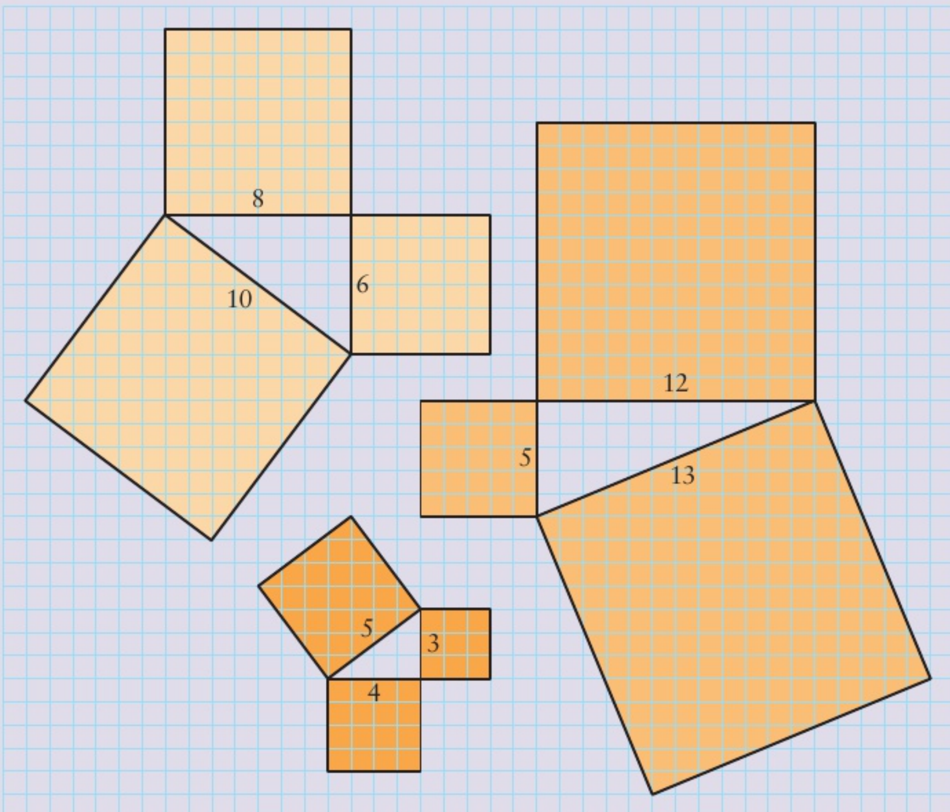


Explore 4.3

For each triangle in the diagram, fill out the following table. In the table, a is the length of the shortest side of the triangle, and A_1 is the area of the square with side length a. The length of the longest side of the triangle is c, and A_3 is the area of the square with side length c.

a	b	c

A_1	A_2	A_3

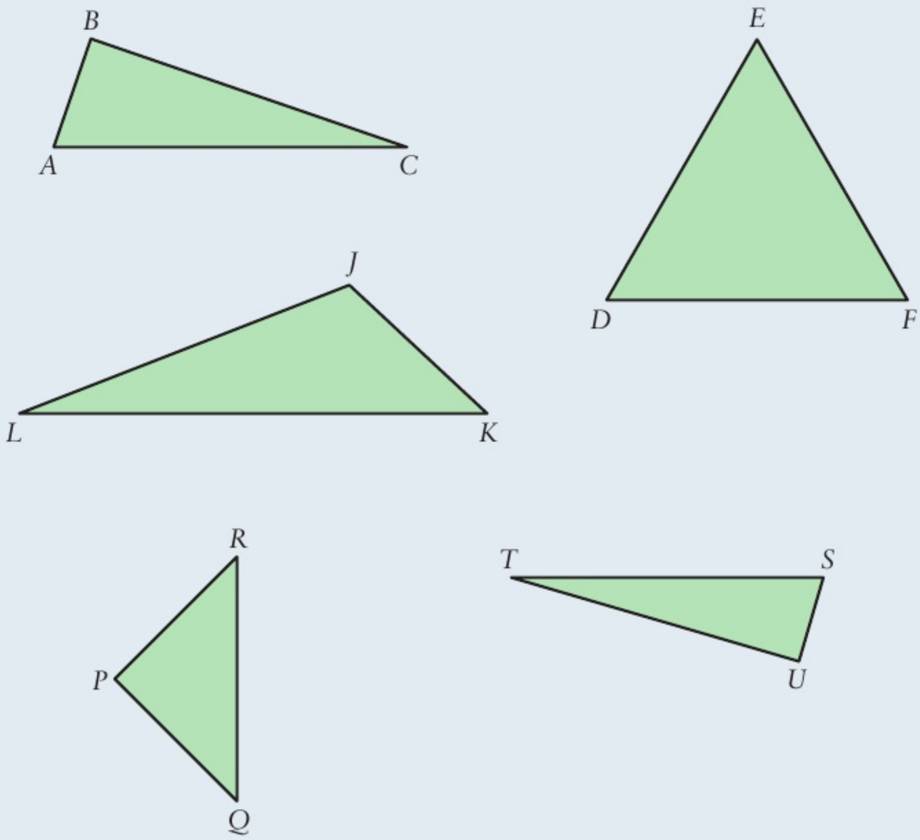


a What relationship do you notice between A_1, A_2 and A_3?

b Hence, what relationship does this suggest between a, b and c?

Practice questions 4.1

1 Which of the following appear to be right-angled triangles?



2 For each right-angled triangle in question 1, identify which side is the hypotenuse.

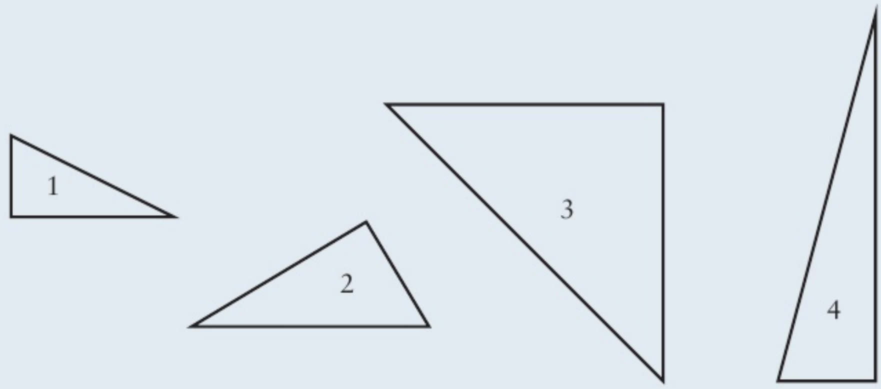
3 For each right-angled triangle in question 1, measure the lengths of the sides in mm.

a Complete the table using your measurements.

Triangle name	(Length of shorter leg)^2	(Length of longer leg)^2	(Length of hypotenuse)^2	(Length of shorter leg)^2 + (Length of longer leg)^2

b What do you notice?

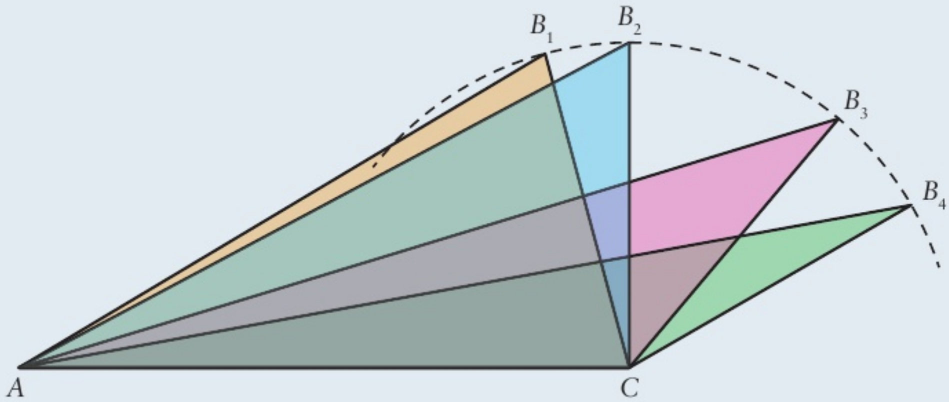
4 Using a ruler, measure the lengths of the legs and hypotenuse of each triangle. Then draw two right-angled triangles of your own on paper or using dynamic geometry software. Write your measurements in the table correct to the nearest mm.



Triangle	Shorter leg, a (mm)	a^2	Longer leg, b (mm)	b^2	Hypotenuse, c (mm)	c^2
1						
2						
3						
4						
5 Construct your own						
6 Construct your own						

What do you notice about the squares of the lengths of the sides of each right-angled triangle?

5 In the diagram below, four triangles are drawn with base AC, which is 7.5 cm long. The points B_1, B_2, B_3 and B_4 all lie on a circle with centre C and radius 4 cm.



Measure the lengths of the sides of the different triangles to complete the table.

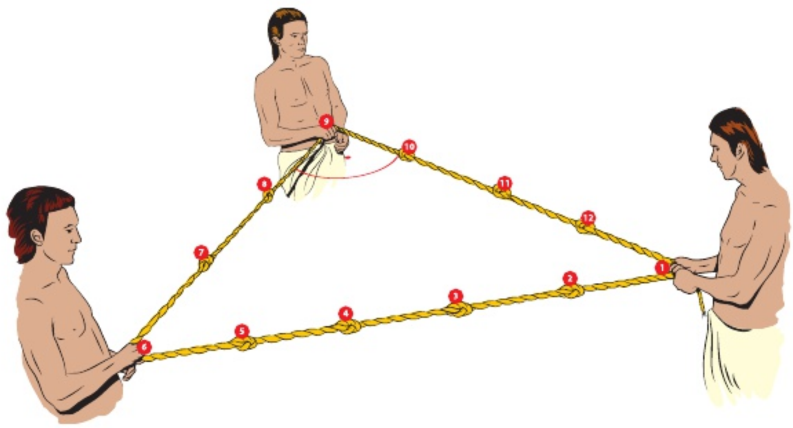
	AC^2	BC^2	AB^2
ΔAB_1C			
ΔAB_2C			
ΔAB_3C			
ΔAB_4C			

a For which triangle does $AC^2 + BC^2 = AB^2$?

b What is the size of angle ACB in this triangle?

4.2 Pythagoras' theorem

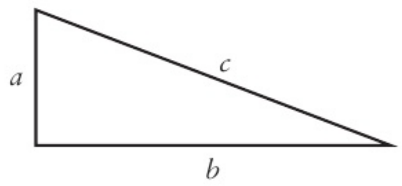
4.2.1 Introducing Pythagoras' theorem



Ancient Egyptians tied equally-spaced knots in a rope to create right angles for construction or marking boundaries.

Draw a right-angled triangle with legs exactly 3 cm and 4 cm. Measure the hypotenuse as accurately as you can. What length did you get?

Pythagoras' theorem states that, in a right-angled triangle, the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse.



If the legs have lengths a and b, and the hypotenuse has length c, then Pythagoras' theorem can be written as

$a^2 + b^2 = c^2$

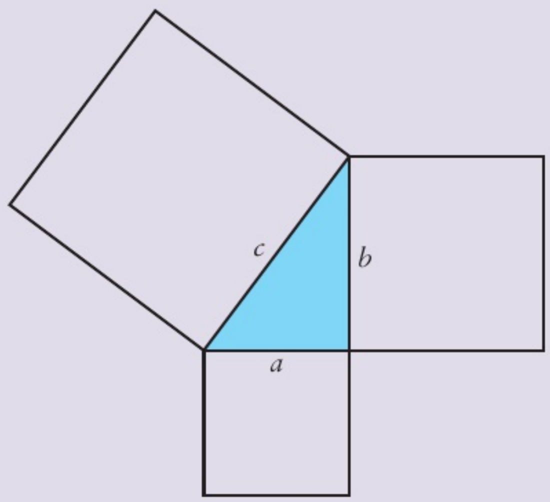
So, the triangle you just drew should have a hypotenuse of 5 cm, since $3^2 + 4^2 = 9 + 16 = 25 = 5^2$

This theorem should also *seem* true from the examples you measured in 4.1, but to prove the theorem, we need to show that it works in the general case, without specific values for the lengths.

4.2.2 Two visual proofs

Explore 4.4

On a plain piece of paper, draw a right-angled triangle with squares on each of its sides as shown in the diagram. Make your drawing as large as possible.

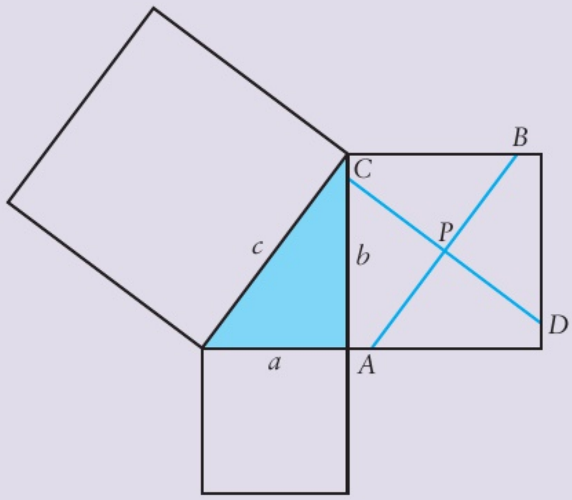


Draw lines on the square on side b as described here:

P is the centre of the square.

AB goes through P and is parallel to the hypotenuse.

CD goes through P and is perpendicular to AB.



Cut out the two smaller squares.

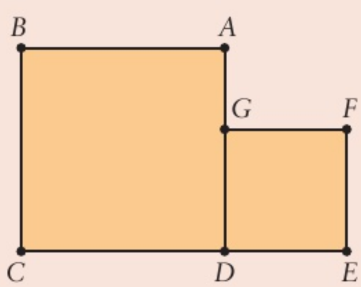
Cut the square on side b into four pieces as shown by the lines you have drawn.

You now have five pieces of a puzzle.

Work out how to put these pieces together to fill the area of the square on side c. What does this tell you?

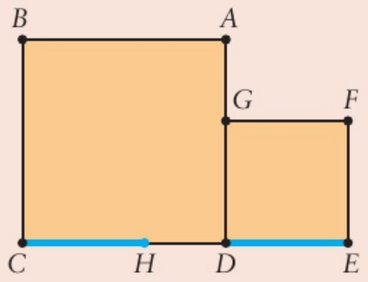
Investigation 4.1

On plain paper, draw a larger square $ABCD$ and a smaller square $DEFG$ side by side as shown. Draw them such that the side length of $DEFG$ is no smaller than 6 cm.

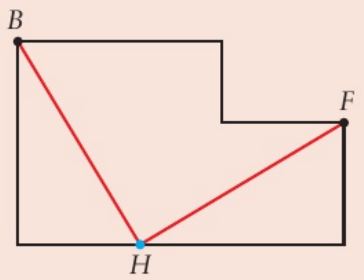


Label the width of the larger square b and the width of the smaller square a.

Measure a units along the base of the larger square from vertex C, and mark point H.



Draw the line segments BH and FH, and cut out the three shapes as shown here:



This is a puzzle! Try to fit the three shapes together to form a single square. If you succeed, you have proved Pythagoras' theorem. Why?

There is one problem with proofs like this: just because we drew and cut as carefully as we could, and just because what we pieced together *looks* like a square, it does not mean it is. Do all the sides have the same length? Are all the angles right angles? Verify that you have, in fact, created a larger square with area c^2.

4.2.3 Finding missing lengths of a right-angled triangle

Let's use Pythagoras' formula to find the length of a side of a right-angled triangle when we already know the lengths of two sides.

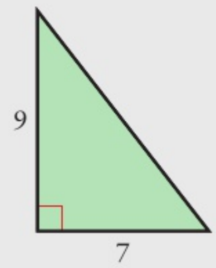
Worked example 4.1

Find the length of the hypotenuse of a right-angled triangle with legs measuring 7 cm and 9 cm. Give your answer correct to 3 significant figures if it is not an exact value.

Solution

Sketching the triangle helps us get an estimate of the expected answer and also helps us avoid careless mistakes.

Substituting into the formula $a^2 + b^2 = c^2$ we have



$7^2 + 9^2 = c^2$

$49 + 81 = c^2$

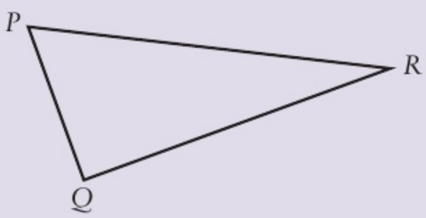
$c^2 = 130$

To get the length of the hypotenuse c, we take the square root:

$c = \sqrt{130} \approx 11.4\,\text{cm}$ (3 s.f.)

Explore 4.5

In triangle PQR, $PQ = 0.4$, $QR = 0.9$, and PQR is a right angle.
Find PR.



A student has answered this question but has made some mistakes. Can you correct the solution they have given? Write a sentence stating in which step(s) of Polya's problem-solving process the mistake occurred (Understand the problem, Make a plan, Carry out the plan and/or Look back) and why.

Solution

Since PQR is a right-angled triangle and we know two sides, we can use Pythagoras' theorem to find the third side.

$a^2 + b^2 = c^2$

$PQ^2 + QR^2 = PR^2$

$0.4^2 + 0.9^2 = 0.16 + 0.81 = 0.97$

Hence, $PR = 0.97$

Worked example 4.2

A fire engine with an aerial platform is used to rescue the residents of a burning building. The base of the ladder sits on the fire engine at a height of 2.5 m off the ground, and the ladder can extend to a maximum length of 30 m.

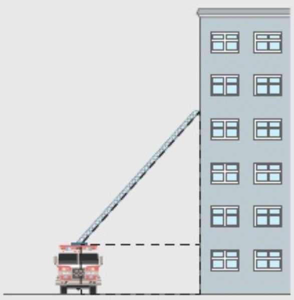
a If the fire engine is parked on the road, 9.5 m in front of the building, what is the maximum height on the building that the aerial platform can reach?

b If each floor of the building is approximately 4.25 m high, what is the highest floor that can be reached using this ladder?

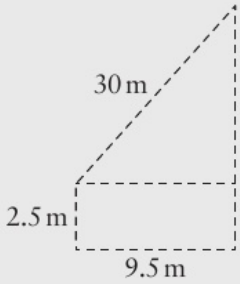
Solution

We need to find how far up the side of the building the aerial platform will reach.

Draw a diagram showing the known measurements and those we wish to find.



Just looking at the shapes involved gives us this diagram:



a We need to find the height of a right-angled triangle, and we know the lengths of the base and hypotenuse, so we can use Pythagoras' theorem to substitute the values we know, then rearrange the formula to find the length of the unknown leg.

$a^2 + b^2 = c^2$, where $a = 9.5$ and $c = 30$

Therefore, $9.5^2 + b^2 = 30^2$

$90.25 + b^2 = 900$

$b^2 = 809.75$

$b = \sqrt{809.75} \approx 28.5\,\text{m}$ (3 s.f.)

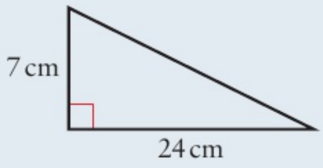
Reading the question again, we see this is not the answer to the question. The maximum height of the platform above ground is the height of the rectangle plus the height of the triangle,
$2.5 + 28.5 = 31.0\,\text{m}$

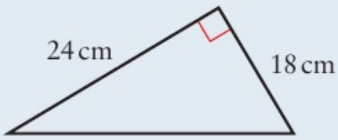
b To answer the second question, we divide this height by the height of each storey and round the result.

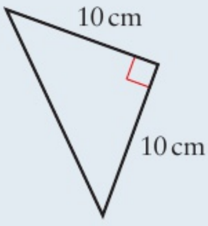
$\frac{31.0}{4.25} \approx 7.3$, so the aerial platform can be used to reach people as high as 7 storeys.

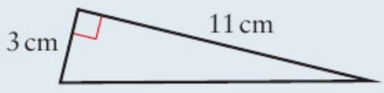
Practice questions 4.2.3

1 Find the length of the hypotenuse of each triangle. Give your answer correct to 3 significant figures where necessary.

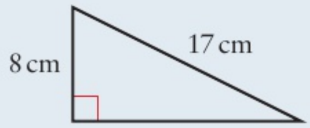
a


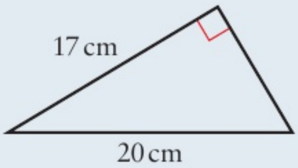
b


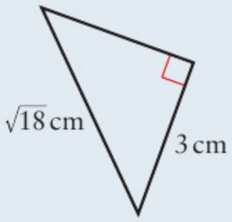
c


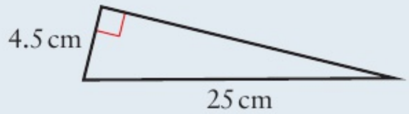
d


2 Find the length of the missing leg of each triangle. Give your answer correct to 3 significant figures where necessary.

a


b


c


d


3 Using square paper, we can accurately draw a line segment of length $\sqrt{5}$ cm by drawing a triangle of height 1 cm and base 2 cm. Since $1^2 + 2^2 = 5$, the hypotenuse's length is $\sqrt{5}$. Using this method, accurately draw a line segment of length:

a $\sqrt{10}$ cm

b $\sqrt{26}$ cm

c $\sqrt{53}$ cm

d $\sqrt{89}$ cm

4 Given the lengths of two sides of a right-angled triangle, find the length of the third side if it must be an integer value.

a 4 and 5

b 5 and 12

c 1.75 and 6.25

d 37.5 and 42.5

5 **a** Children are dancing around a 3.5 m maypole. The ribbons they are holding, which are tied to the top of the pole, are 4 m long. How far can they stand from the maypole at the start of the dance? (What assumptions are you making to solve this problem?)

Fact Q5b

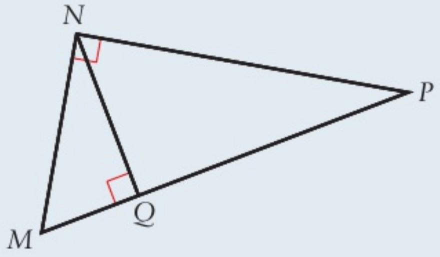
The **altitude** of an object is its height above a surface, usually the ground below it or sea level.

b An airplane is ascending in a straight path after take-off. When the plane has flown a distance of 30 km, its position on a map is 28 km from the airport. What is its altitude?

6 **a** Find the length of the diagonal of a rectangle with dimensions 13 cm by 18 cm.

b Find the height of an equilateral triangle whose perimeter is 20 cm.

7 In this diagram $MN = 60$ cm and $NQ = 48$ cm. Find the length of MP.

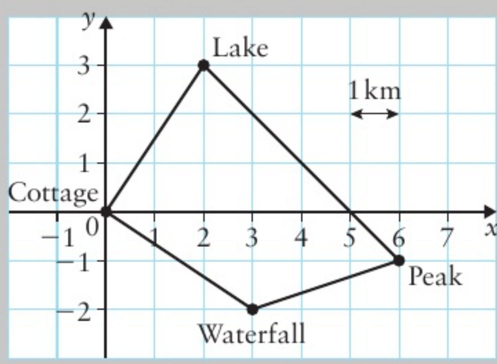


4.2.4 Distances on the Cartesian plane

How can we use Pythagoras' theorem to find the distance between two points?

Worked example 4.3

A group of hikers walk from their cottage to see a waterfall and mountain peak, then they have a swim at the lake before returning to the cottage.
The map shows the relative position of each location.
The scale of the map is 1 unit : 1 km.



Find the total distance that they walk.
Give your answer to the nearest 100 m.

Solution

Understand the problem

We need to find the perimeter of the quadrilateral representing their journey, so we need to find the length of each side.

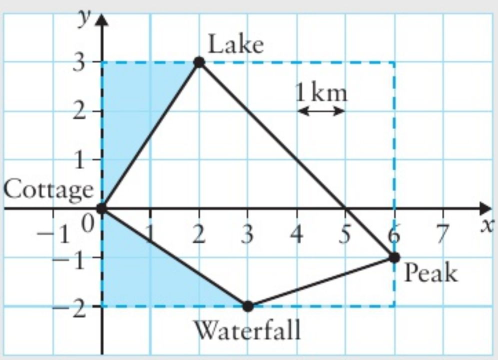
Make a plan

Since the segments are diagonal lines, we need to find right-angled triangles for which these segments are the hypotenuses. Then we can use Pythagoras' theorem to find their lengths.

Carry out the plan

This diagram shows four right-angled triangles on the outside of the quadrilateral.

We can call the locations C (Cottage), W (Waterfall), P (Peak) and L (Lake).



Using $c^2 = a^2 + b^2$, we are trying to find c, the length of the hypotenuse, in each case.

$CW^2 = 2^2 + 3^2 = 13$ gives $CW = \sqrt{13} \approx 3.61\,\text{km}$

$WP^2 = 3^2 + 1^2 = 10$ gives $WP = \sqrt{10} \approx 3.16\,\text{km}$

$PL^2 = 4^2 + 4^2 = 32$ gives $PL = \sqrt{32} \approx 5.66\,\text{km}$

$LC = CW$, since the shaded triangles are rotations of one another.

Now we can add the individual distances to find the perimeter, then change the answer to metres.

The perimeter is $2\sqrt{13} + \sqrt{10} + \sqrt{32} \approx 16.0\,\text{km}$, or 16 000 m, to the nearest 100 m.

Connections

Transformations: Can you identify the centre of rotation?

Fact

One mile is approximately 1609.344 m. How much further did the hikers need to walk to reach 10 miles?

Investigation 4.2

1. Plot the points $A(-3, 1)$ and $B(5, 6)$ on square paper. Choose an appropriate size for your axes before drawing, and consider using a scale of 1 unit : 1 cm.
2. Draw the line segment AB.
3. With a scale of 1 unit : 1 cm, estimate the distance between A and B by measuring with a ruler. Your value will only be an estimate, since the true distance is an irrational number, and you may have inaccuracies in your measurement from drawing and/or using the ruler.
4. Draw a right-angled triangle using AB as the hypotenuse.
5. Use Pythagoras' theorem to find the exact length of AB in surd form (radical or root form), then use a calculator to find the length correct to 3 significant figures. How close was your estimate?
6. Work out the percentage error using the formula:

$$\text{percentage error} = \frac{\text{measured value} - \text{actual value}}{\text{actual value}} \times 100\%$$

7. What does it mean if your percentage error is positive?
8. What does it mean if your percentage error is negative?
9. Repeat the process for the distance between points $C(1, -1)$ and $D(2, -3)$.

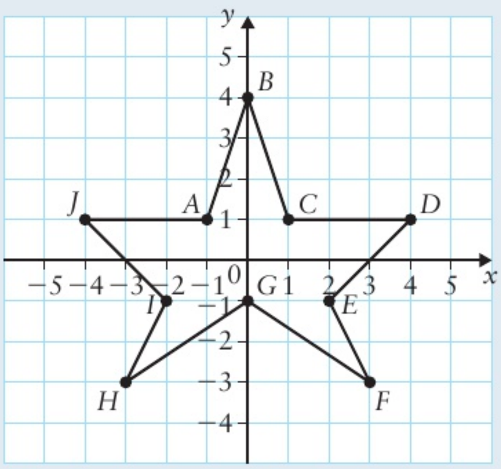
10 Was your measured estimate the same amount from the actual distance as the results for AB?

11 How did your percentage error compare this time?

12 Why might your percentage error be quite different, even if the size of your error was similar?

Practice questions 4.2.4

1 Find the perimeter of this shape.



2 For each pair of points, plot them on a coordinate grid and find the distance between them.

a (2, 5) and (6, 2)

b (−4, 0) and (5, 5)

c (0, 1.5) and (−7, −1)

3 The vertices of a quadrilateral are given. Find the perimeter of the shape.

a (2, 5), (6, 2), (1, −10) and (0, 0.5)

b (−3, −8), (−10, 5), (2, 6) and (7, −4)

4 If you know the lengths of the three sides of a triangle, you can test to see if it is a right-angled triangle using Pythagoras' theorem. If the squares of the lengths of the two shorter sides sum to the square of the length of the longest side, it must be a right-angled triangle.

For example, a triangle with sides 39, 80 and 89 is a right-angled triangle:

$39^2 + 80^2 = 7921 = 89^2$

However, 11, 19 and 22 are not the sides of a right-angled triangle:

$11^2 + 19^2 = 482 \neq 484 = 22^2$

Given the vertices of the following triangles, determine which of them are right-angled.

a $(-2, 5)$, $(6, -10)$ and $(-2, -10)$

b $(1, 1)$, $(8.5, 11)$ and $(25, -17)$

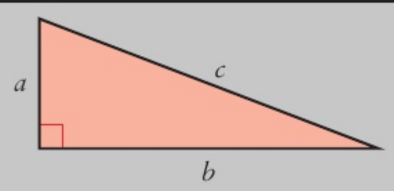
c $(-7, 5)$, $(-3, 2)$ and $(-11, -4)$

4.3 More proofs and problems

4.3.1 Other proofs

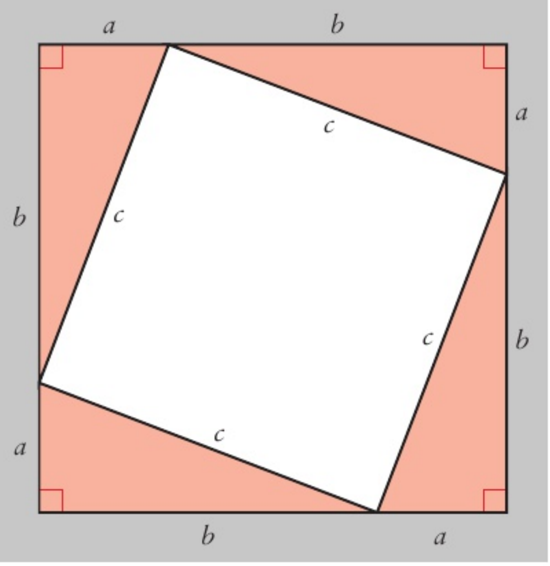
Worked example 4.4

Let the length of the shorter leg of a right-angled triangle be a, the length of the longer leg be b, and the length of the hypotenuse be c.



Four of these triangles can form a square, as shown.

Prove Pythagoras' theorem using areas and the diagram.



Solution

1 The four right angles of the triangles mean that the large outer shape is a rectangle. Each side of the rectangle has length $a + b$, hence the rectangle is a square. The area of this square is

$$\begin{aligned}(a + b)^2 &= (a + b)(a + b)\\ &= a^2 + ab + ab + b^2\\ &= a^2 + 2ab + b^2\end{aligned}$$

2 Now express the area of the large square as the sum of the smaller shapes inside.

Each side of the inner square is the hypotenuse of one of the right-angled triangles, therefore its area is c^2. Each right-angled triangle has base b and height a, so each has area $\frac{1}{2}ab$

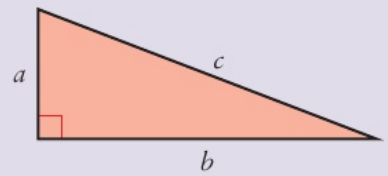
3 Set the two expressions equal to each other, and simplify.

The area of the inner square and four triangles is $c^2 + 4 \times \frac{1}{2}ab = c^2 + 2ab$, and this is the same as the area of the outer square.

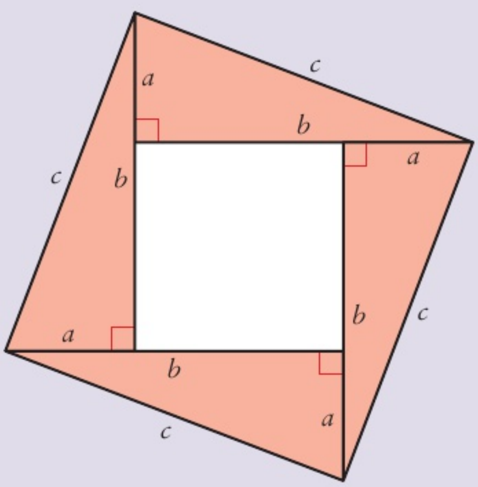
Therefore, $a^2 + 2ab + b^2 = c^2 + 2ab$. Subtracting $2ab$ from each side gives the desired result: $a^2 + b^2 = c^2$

Explore 4.6

Let the length of the shorter leg of a right-angled triangle be a, the length of the longer leg be b, and the length of the hypotenuse be c.



Four of these triangles can form a square as shown.



Can you prove Pythagoras' theorem using areas and the diagram? Here is a suggestion of the steps you could follow:

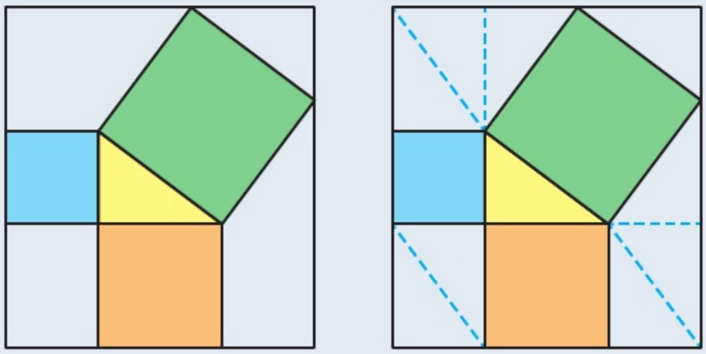
1 Express the area of the small, inner square in terms of a and b.

2 Express the area of the small, inner square as a difference of the outer shapes.

3 Set the two expressions equal to each other, and simplify.

Practice questions 4.3.1

1 Complete the steps of the following proof.



The side length of the blue square is a units, the side length of the orange square is b units, and the side length of the green square is c units.

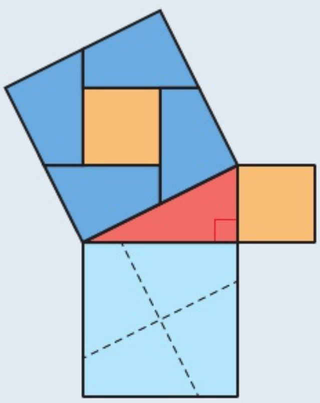
a The base of the large rectangle is ________ units, and its height is ________ units. Therefore, its area is ________, which expands to ________.

b There are ___ congruent right-angled triangles inside the rectangle, each with area ________. Hence, the coloured area is ________, which simplifies to ________.

c ____, so we can conclude that $c^2 = a^2 + b^2$

2 The diagram here is called Perigal's dissection, named after Henry Perigal (1801–1898), a British stockbroker and amateur mathematician.

The dashed lines in the light blue square are parallel to the sides of the dark blue square, and they meet in the centre of the light blue square.



Challenge Q2

Fact Q2

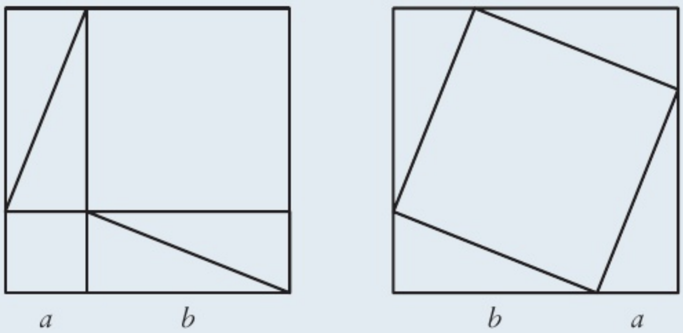
Perigal was so proud of this proof that he had the diagram engraved on his tombstone.

The sides of the orange square (inside the dark blue square) are parallel to the legs of the red triangle and are extended to meet the midpoints of the sides of the dark blue square.

Show that the orange square is congruent to the square on the shorter leg of the red triangle (and hence, that this dissection is a proof of Pythagoras' theorem). Feel free to create labels for vertices or lengths to make your description easier for others to understand.

3 The diagram on the left shows a large square with base $a + b$ that contains two smaller squares with bases a and b respectively.

The diagram on the right shows another large square with base $a + b$ that contains a smaller square whose vertices are on the perimeter of the large square, a units from the nearest vertex of the large square.



a Find the area of the smaller square in the diagram on the right in terms of a and b.

b Explain how this proves Pythagoras' theorem.

Reflect

You have seen and completed several different 'proofs' of Pythagoras' theorem.

What is the value of finding many different solutions to a problem?

4.3.2 Miscellaneous exercises involving Pythagoras' theorem

Worked example 4.5

Sasha sails in Lake Neusiedler in Austria in a straight line from Podersdorf to Rust, which is 6.4 km to the South and 9.7 km to the West of Podersdorf. She sails at a constant speed of 9 knots. How long does it take her to get to Rust, to the nearest minute?

Hint

The key to solving worded problems in geometry is to complete Step Zero: draw a diagram! For each of the exercises in this section, draw a diagram that fits the situation described in the problem before attempting to solve.

Fact

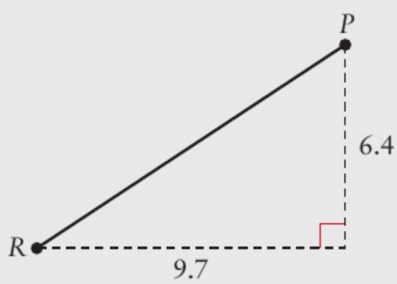
1 knot ≈ 1.852 km/h

Solution

For the distance

We are given the speed and need to find the time travelled, so first we will work out the distance travelled.

First, we will draw a diagram to see what length is missing.



Since South and West are perpendicular directions, we have a right angle between the distances of 9.7 and 6.4, meaning that we want to find the length of the hypotenuse *PR*.

We will use Pythagoras' theorem, $a^2 + b^2 = c^2$, to find PR.

$PR^2 = 9.7^2 + 6.4^2$

$PR = \sqrt{9.7^2 + 6.4^2} \approx 11.6\,\text{km}$ (3 s.f.)

For the time

Since the distance is given in km, we need to change our speed units to km/h.

We know that 1 knot ≈ 1.852 km/h, and using the distance–speed–time triangle, we know that time = $\frac{\text{distance}}{\text{speed}}$, so we can use this information to find the time travelled.

Sasha's speed is 9 knots = $9 \times 1.852 = 16.668\,\text{km/h}$

Her journey time is $\frac{11.621}{16.668} \approx 0.697\,\text{h}$

$0.697 \times 60 \approx 41.8$ minutes (3 s.f.)

Re-reading the question, we need to give the time to the nearest minute, which would be 42 minutes.

This seems reasonable using estimation: the base and height of the triangle are approximately 10 and 6, respectively, which would mean the hypotenuse is close to $\sqrt{100 + 36} = \sqrt{136} \approx \sqrt{144} = 12$.

10 knots would be approximately 18.5 km/h, and 12 is about two-thirds of this, so it should take around two-thirds of an hour, or 40 minutes to complete the trip.

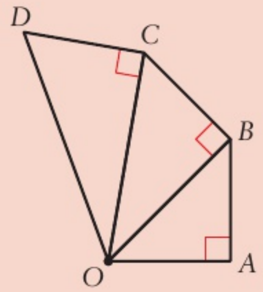
Reflect

Why is it useful to draw a diagram, even when we might find a problem straightforward and easy to understand?

Challenge

Investigation 4.3

Copy and continue the Pythagorean spiral diagram shown here, using the rules that follow.



Start with the isosceles, right-angled triangle *OAB*, which has a base and height of 1.

Each subsequent right-angled triangle is drawn using the previous triangle's hypotenuse as a base, and the height of each new right-angled triangle is 1.

So, $OA = AB = BC = CD = DE = \ldots = 1$

Draw as many more triangles as you can without overlapping triangles. That is, do not draw a triangle that would cover all or part of triangle *OAB*. Stop just before this happens.

How many triangles have you drawn?

What is the length of the hypotenuse of *OAB*? Hence, what is the length of the hypotenuse of *OBC* and all the other subsequent triangles?

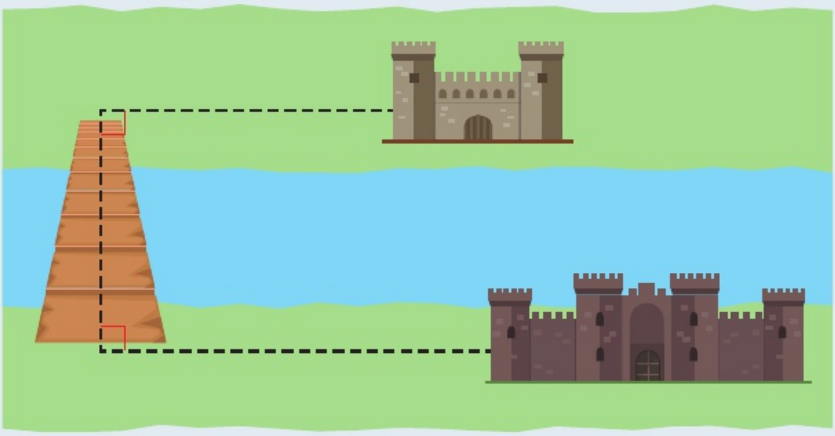
Without drawing them, state at least two ways you could draw a line segment exactly $\sqrt{18}$ units long.

Practice questions 4.3.2

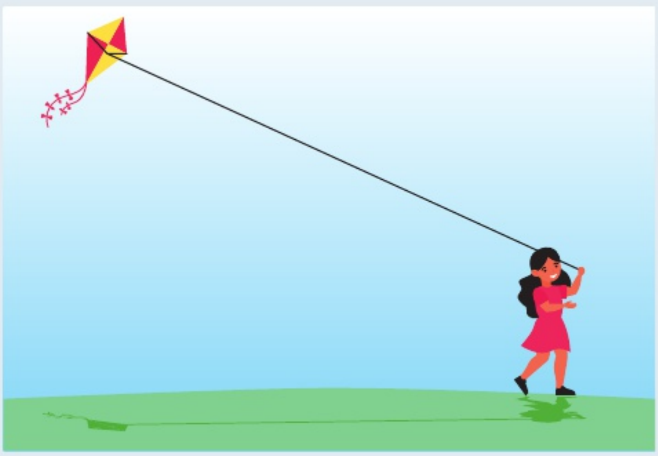
1 In an orienteering competition, the second checkpoint is 1200 m East and 800 m North of the first, and the third checkpoint is 950 m Southeast of the second.

- a Draw a diagram with right-angled triangles showing the situation.
- b How far is the second checkpoint from the first?
- c How far East of the second checkpoint is the third?

2 On a tour of a medieval town, you walk from the fort on one side of the river to the castle on the opposite bank. The fort is 450 m from the bridge, the bridge is 370 m long, and the castle is 600 m from the bridge. What is the direct distance from the fort to the castle?



3 You are flying a kite on a windy day. Your 40 m kite string is fully extended, and the shadow of the kite is directly below it, 18 m away from you. How high is the kite in the air?



4 You need to repair the side of a building. You have a 3-metre long ladder that you place 1 m from the building. How high up the wall will the ladder reach?

5 Your family purchases a new, 55-inch, flat-screen television with an aspect ratio of 16 : 9. The size of a television is the length of the diagonal of the screen, and the aspect ratio is the ratio of the base of the screen to its height.

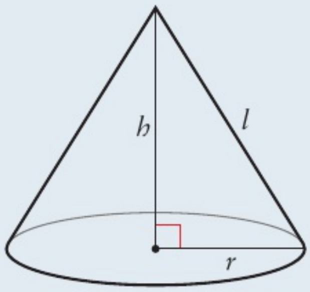
a How wide is the TV in inches?

b What are the dimensions of a 6.5-inch smartphone in centimetres with the same aspect ratio?

Fact Q5

1 inch ≈ 2.54 cm

6 What is the slant height, l, of a cone with diameter 7 cm and height 15 cm?



7 The picture at the beginning of the chapter is of a wooden house by a lake. The wooden construction is in the shape of an isosceles triangle. Looking at the house from the front, the width is 3 m and the height is 4.5 m. The house is 6 m deep.

a Calculate the surface area of the roof to the nearest m^2.

b If the roof is covered with sheet metal whose price is €3.15 per m^2, what is the cost of covering the roof?

8 Krivǻn is the most iconic mountain in Slovakia.

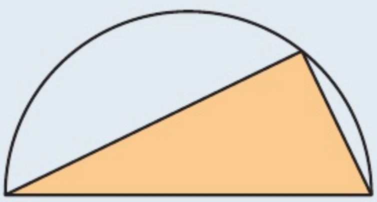
The ridges leading to its peak almost form a right angle.

Kristina and Laura are starting their final approach to the summit of Krivǻn, both at an altitude of 2000 m, but from different ridges 1.36 km from one another as the crow flies.

a According to her map, Kristina is 580 m from the peak, horizontally, but the peak has an elevation of 2495 m. Assuming the ridge has a uniform gradient, how far must she climb to the summit?

b Hence, how far does Laura need to hike to reach the top, assuming the two ridges meet at a right angle?

9 When a triangle is constructed inside a semicircle, with the diameter as its hypotenuse, it is always a right-angled triangle! (Get a pair of compasses and a protractor, and test it out.)

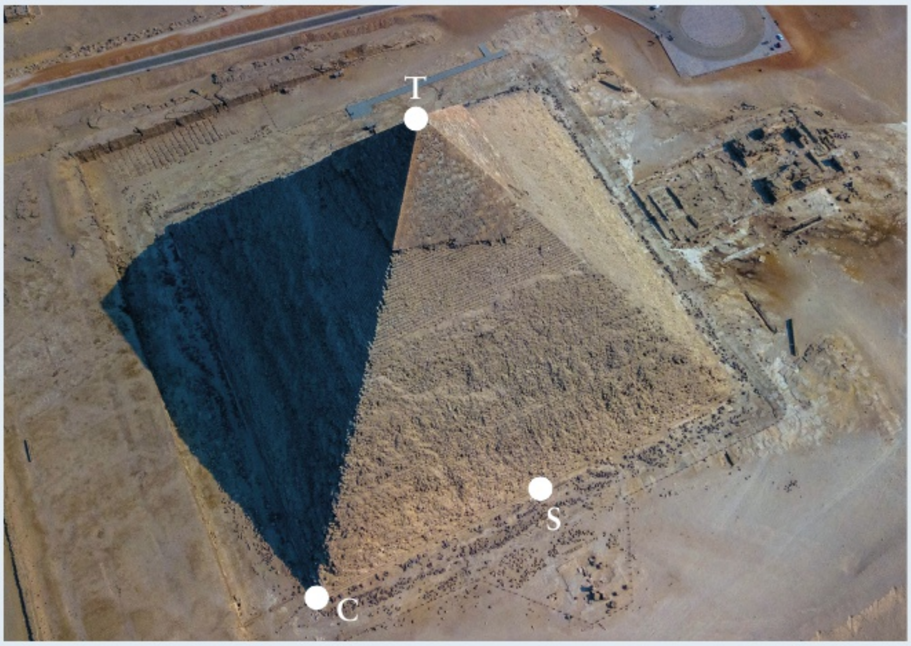


Find the area of all triangles that can be constructed in a semicircle of radius $\sqrt{21.25}$ cm for which both legs have integer lengths.

Connections

Geometry of circles / Circle theorems

10 The Great Pyramid at Giza in Egypt is a square-based pyramid. It has eroded over time, but the original length of its base was 440 Egyptian royal cubits, and its height was 280 cubits.



If you walked up to the top of the pyramid (point T) when it was built, what distance did you travel (in cubits) if:

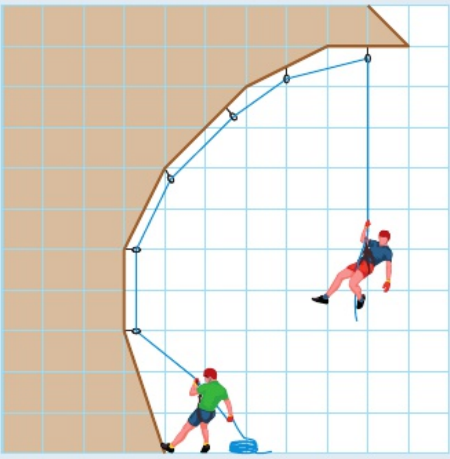
a you walked up from point S, the midpoint of the side of the base of the pyramid?

b you walked up from point C, the corner of the base of the pyramid?

Fact Q11

A belayer is a person who controls the safety rope for a climber.

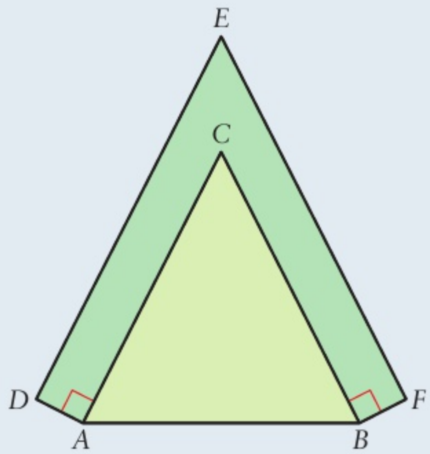
11 A climber is lowered down by her belayer after finishing an overhanging route as shown in the diagram, which is overlaid on a grid where each square is 1 m by 1 m.



Assume the rope goes from the foot of the climb by the belayer, along the wall to the top of the climb, and down to the climber on the ground (after being lowered completely). To the nearest metre, how much rope is needed to safely complete this climb?

12 In the diagram, the base of triangle ABC is $AB = 6\,\text{cm}$, and its height is $8\,\text{cm}$. Also, AC is parallel to DE, and the line through CE is a line of symmetry for the entire diagram.

Challenge Q12



a Find the length of AC.

b If $AD = 3\,\text{mm}$, find the length of DE.

c Find the area of hexagon $ACBFED$.

Self assessment

- I can square numbers and estimate square roots.
- I can round numbers to 3 significant figures.
- I can identify the hypotenuse of a right-angled triangle.
- I can use Pythagoras' theorem to find the length of the hypotenuse of a right-angled triangle.
- I can use Pythagoras' theorem to find the missing length of a leg of a right-angled triangle.
- I can construct a diagram from information given in a worded problem.
- I can apply Pythagoras' theorem to solve worded problems.
- I can complete the steps of a proof when the structure of the proof is given.
- I can use logical reasoning to complete a proof without being given its structure.

Check your knowledge questions

1 On plain paper, draw four congruent, non-isosceles, right-angled triangles and cut them out.

a Piece them together so that there is a square-shaped hole in the centre of the four puzzle pieces.

b Find a second method to piece them together. Is the inner square (the hole) the same size as in the first solution? Is the outer square the same size as in the first solution?

Hint Q1a

A larger square should be formed by the outer edge of the four puzzle pieces.

2 The lengths of the legs of a right-angled triangle are given. Find the length of the hypotenuse in each case. Draw a sketch of each triangle.

a 5 and 12 b 8 and 6 c 1.4 and 4.8

d 4 and $\sqrt{7}$ e $\sqrt{17}$ and $\sqrt{47}$ f 10 and 10

3 The length of the hypotenuse and one leg of a right-angled triangle are given. Find the length of the missing leg. Draw a sketch of each triangle.

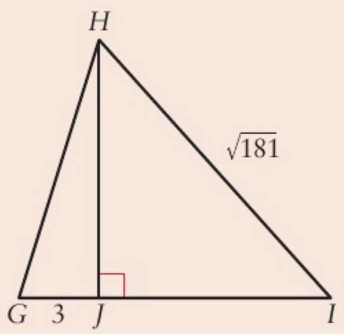
a 9 and 15 b 6.5 and 2.5 c 8 and 9

d 1.1 and 2 e 8 and $4\sqrt{3}$ f $\sqrt{3}$ and $\sqrt{6}$

4 Find the length of the diagonal of a rectangle measuring 5.6 cm by 87 mm.

5 Find the height of an isosceles triangle with base 9 cm and congruent sides of length 5 cm each. Hence, find its area.

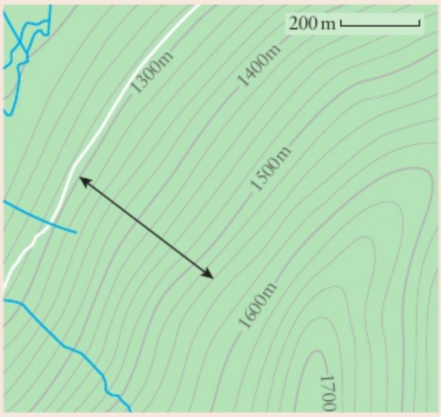
6 In the diagram below, find the exact length of GH (surd form) given that $GI = 12$



7 Find the area of an equilateral triangle of side length 12 cm.

8 A rhombus has diagonals of lengths 20 cm and 37.5 cm.
Find the perimeter of the rhombus.

9 Each pair of contour lines on this map represents a rise between them of 20 m.



How far must you hike along the slope to hike the steep route indicated by the arrow? Assume a constant steepness during the hike. Measure as accurately as possible using the map scale as a guide.

Challenge Q9

10 Road steepness can be shown as a percentage, as on the sign on the right.

A road with 12% steepness goes down 12 m (vertically) for every 100 m it goes horizontally. How far did you drive along the road if your elevation decreased by 20 m?

11 A spider is hanging 1 m below the ceiling of a rectangular gymnasium, 1.5 m from the North wall and 3 m from the West wall. How far is the spider from the corner of the gymnasium where these walls meet the ceiling?

Challenge Q11

Connections

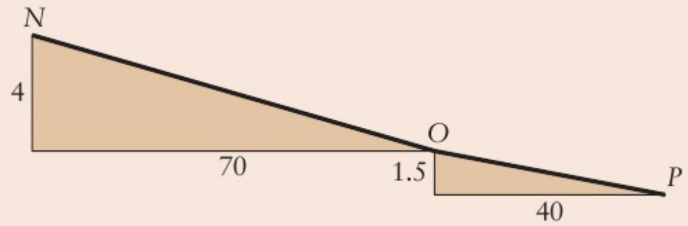
Locus

12 List all the points with integer coordinates that are exactly 25 units from the origin.

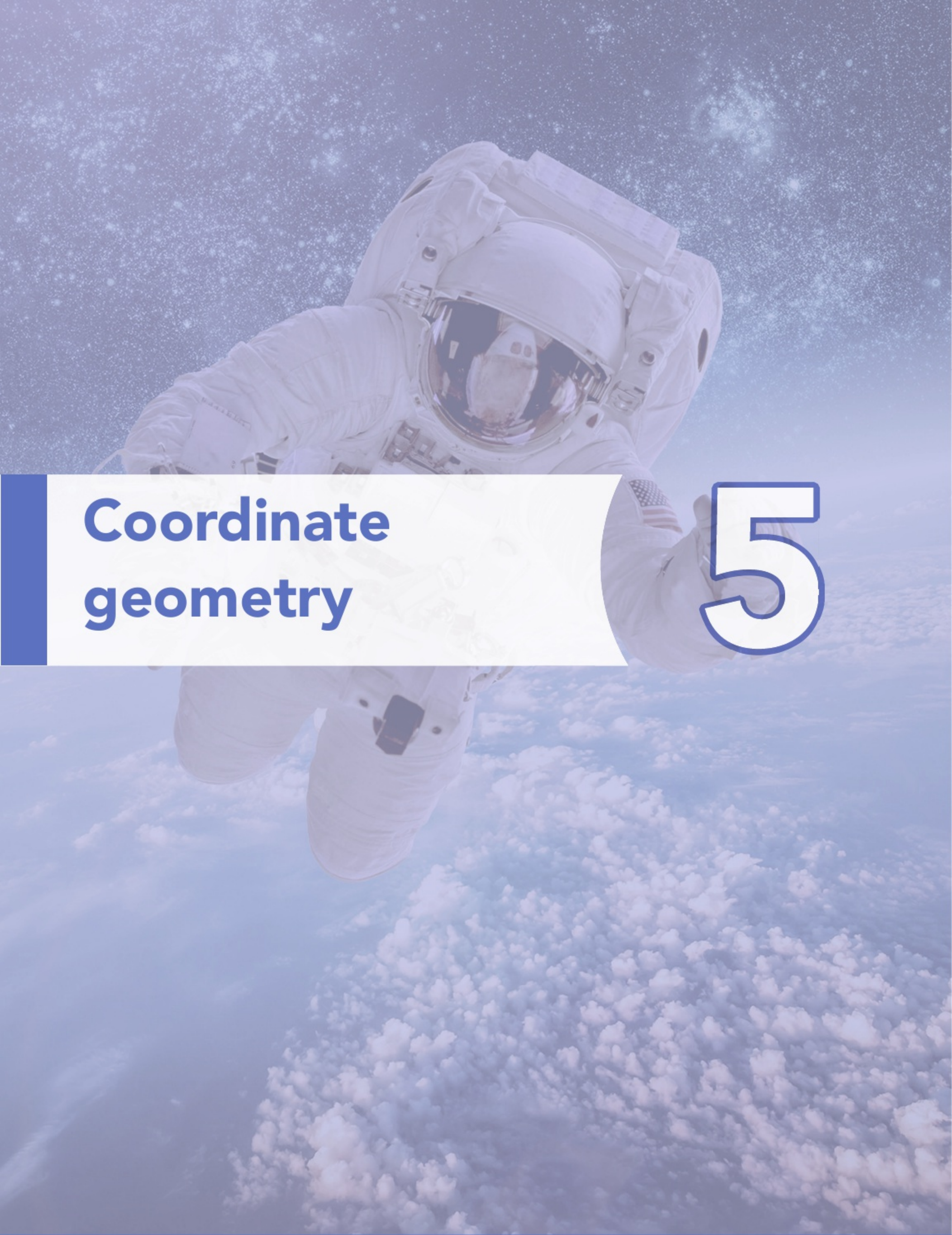
Challenge Q12

13 A ropes course has a cable between two trees which are 30 m apart on flat ground. The starting height on the first tree is 6 m, and it ends at a height of 4 m on the second tree. How long is the cable? Assume the cable is taut.

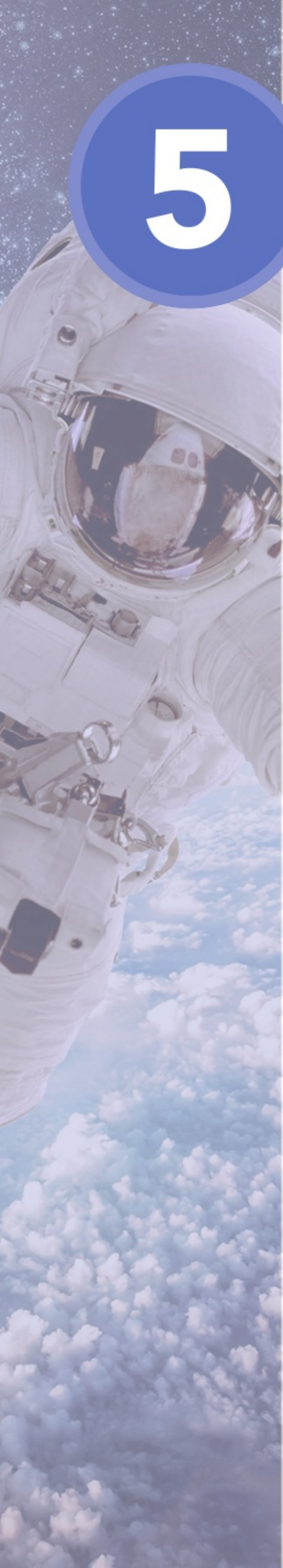
14 A tunnel runs underground from point N to O to P, as shown in the diagram.



The distances shown are in metres. What is the length of the tunnel?



5 Coordinate geometry



5 Coordinate geometry

KEY CONCEPT

Form

RELATED CONCEPTS

Change, Representation, Space

GLOBAL CONTEXT

Orientation in space and time

Statement of inquiry

Forms in space help us to understand changes in representation of objects.

Factual

- What is an ordered pair in a coordinate system?
- What is the gradient of a straight line?

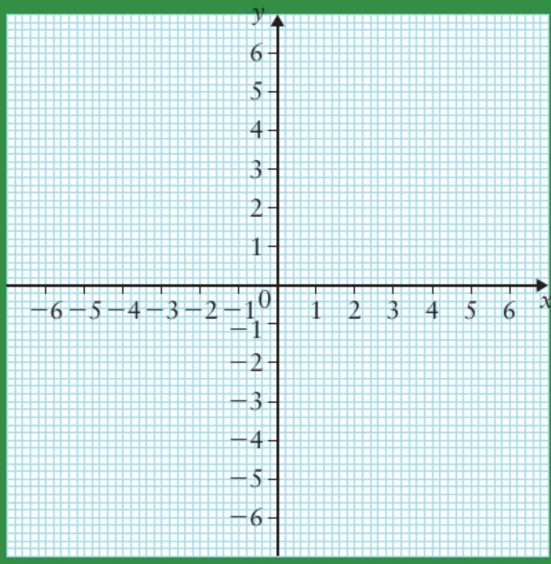
Conceptual

- How can you determine the equation of a straight line?
- How do you know whether two lines will intersect?

Debatable

- Do vertical lines have undefined gradients or no gradients?

Do you recall?



1 Do you remember how to represent a point with coordinates (x, y) in the number plane? Copy the number plane and plot these points: $A(3, 4)$, $B(-2, 3)$, $C(-2, 0)$, $D(5, -1)$

2 **a** Plot the points $A(1, 1)$, $B(5, 1)$ and $C(5, 3)$ and join them to form triangle ABC.

b What is the length of each side of the triangle?

3 Sketch the graph of $y = 3x - 2$

4 Solve each equation

a $\dfrac{x-1}{2} = 3$

b $7x + 3 = 3x + 11$

5 Solve the inequality $2x + 3 > 7$ and graph its solution on a number line.

5.1 Points in the number plane

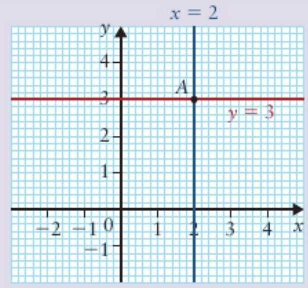
Hint

Ordered pairs are used to describe points in the coordinate plane. For example, the point (4, 3) has x-coordinate 4 and y-coordinate 3.

Explore 5.1

Look at the lines drawn in this diagram.

List all the information that the diagram shows. Include the coordinates of the intersection point, A.



Fact

The x-axis can be described as the line $y = 0$ and the y-axis as the line $x = 0$

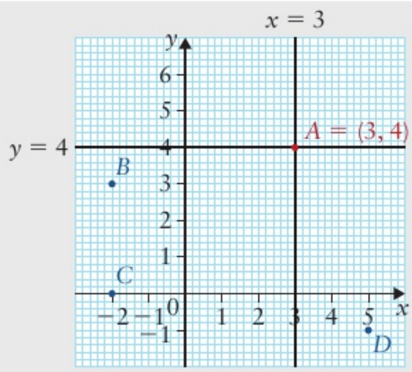
Worked example 5.1

Plot the points $A(3, 4)$, $B(-2, 3)$, $C(-2, 0)$ and $D(5, -1)$ on the coordinate plane.

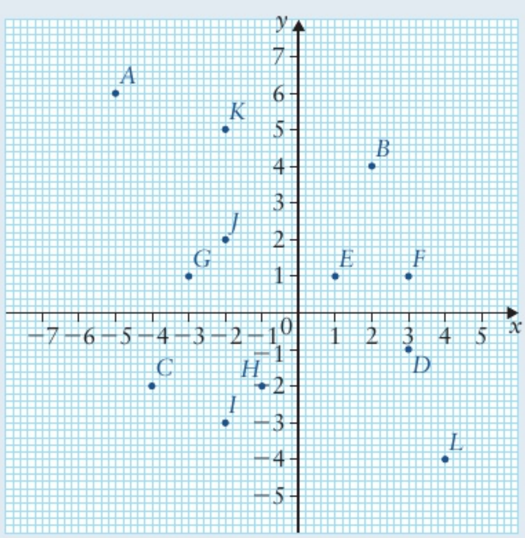
Solution

Point A is the point of intersection of the lines $x = 3$ and $y = 4$

You can find points B, C and D in a similar way.



Practice questions 5.1



1 Use the diagram on the previous page to write the coordinates of each of these points.

a A b B

c C d D

e E f F

g G h H

i I j J

k K

2 Using the diagram in question 1, answer the following questions.

a What type of triangle is EFD? Find its area.

b What type of quadrilateral is $GEHC$? Find its area.

3 Draw a coordinate plane with x- and y-axes from −6 to 6.
Plot each set of points and join them in the order given.
Name each geometrical shape that you make.

a (−2, 2), (2, 2), (2, −2), (−2, −2),(−2, 2)

b (−5, 1), (−2, 1), (0, −1), (−6 ,−1), (−5, 1)

c (0, 1), (2, −2),(−3, −1), (0, 1)

d (−3, 4), (−1, 4), (−2, 2), (−4, 2), (−3, 4)

4 Find the distance between each pair of points.

a (2, 5) and (−3, 5)

b (−1, 4) and (−7, 4)

c (−3, 4) and (−3, −5)

d (4, 9) and (4, −1)

e (2, −2) and (5, 2)

5 a On a coordinate plane, plot the points $A(-2, 3)$, $B(3, 3)$, $C(4, -1)$ and $D(-1, -1)$ and join the points with straight lines in the order given.

b Use what you learned in the previous questions to find the area of $ABCD$.

c Find the perimeter of $ABCD$.

Challenge Q2

Hint Q4a

Plot the points and count the number of units between them.

Hint Q4e

Use Pythagoras' theorem $a^2 + b^2 = c^2$

Challenge Q4e

Challenge Q5

6 Complete the tables by filling in the missing x- or y-coordinates for each of these lines.

Line AB

x	−3		3	
y		3		1

Line CD

x	−3		−1	
y		2		6

Line EF

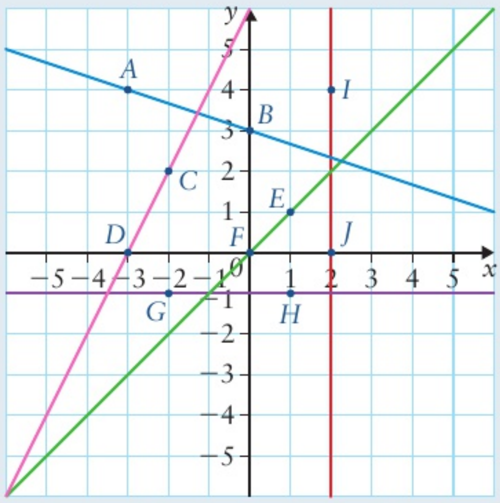
x	1		−3	
y		2		−5

Line GH

x	−4		0	
y		−1		−1

Line IJ

x	2		2	
y		0		−4



Challenge Q7

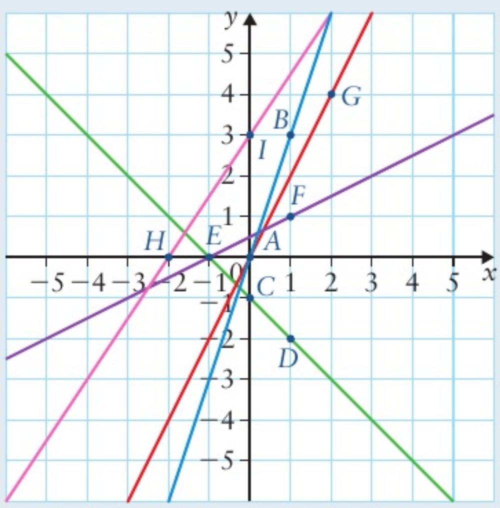
7 Match each of the lines in the grid below with the correct rule.

a The x-coordinate is half of the y-coordinate.

b The sum of the x- and y-coordinates is −1.

c The y-coordinate is three times the x-coordinate.

d The x-coordinate is 1 less than double the y-coordinate.



Connections

Fun with the number plane: sinking battleships!

This is a game for two players/groups.

1 Each player draws their fleet of ships on a coordinate grid by plotting points at the intersection of the gridlines. Agree on the size of the coordinate grid in advance – the bigger it is, the longer the game is likely to take. Each ship must be represented by a continuous row of points along either a horizontal, vertical or sloping line. Each person's fleet should consist of ships with 5, 4, 3 and 2 plotted points.

2 The aim of the game is to 'sink' each other's ships by guessing where they are positioned on the grid.

3 Players take turns to play. Each player is given 2 'shots' on each turn. For each shot you should guess a pair of coordinates where you think your opponent has positioned a ship. The coordinates you try must consist of all combinations of sign, that is, one each of (+, +), (−, +), (−, −), and (+, −). A player who violates this rule forfeits the rest of the turn. You should record the shots that both you and your opponent make on separate charts so you know what has been hit.

4 If a shot misses, your opponent declares 'miss' and you both place an open circle in the appropriate position on your chart.

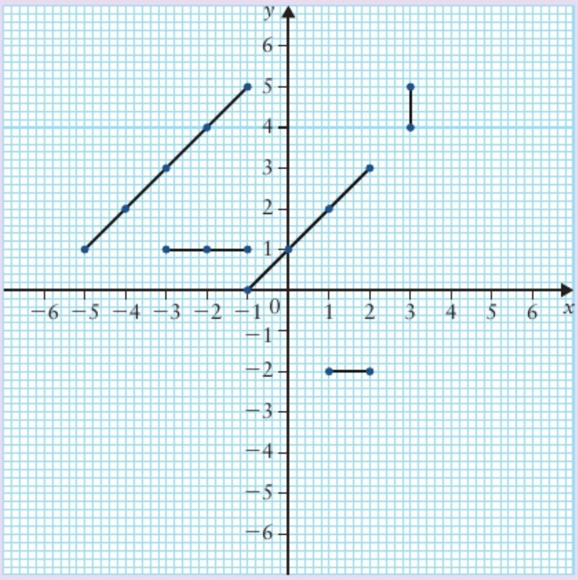
5 If a shot hits, your opponent declares 'hit' and you both place an X in the appropriate position.

6 Your opponent should tell you when you have found all the coordinates of an entire ship, by saying, for example, 'You sank a ship of size 3.'

7 Continue taking turns to play until one of you has sunk the other's entire fleet.

Here is an example of how to draw a fleet of ships.

Guessing (1, −2) is a hit and (0, 0) is a miss!



5.2 Graphing straight lines

Explore 5.2

Imagine you are a carpenter. You have a straight edge, a pencil and a piece of wood. Before cutting the wood, you need to draw a line so that you can see where to use your saw. What do you need to know/do?

Hint

You need at least two points to draw a straight line.

The points on a straight line are linked by an equation. To graph a straight line, you need to plot at least two points on the line that satisfy this equation. Two points that can easily be found are the ***x*-intercept**, where the straight line crosses the x-axis, and the ***y*-intercept**, where the straight line crosses the y-axis.

Worked example 5.2

A straight line has equation $x + y = 4$. The table shows the x- and y-coordinates of the line.

Fill in the missing values in the table.

x	1		3		5
y		2		5	

Plot the points on a coordinate plane and join them to draw the graph of the line.

Solution

Understand the problem

A straight line is a set of points on a coordinate plane. The x- and y-coordinates of each point must satisfy the equation of the line.

Make a plan

We need to substitute the given x- or y- coordinate into the equation of the line to figure out the missing ordered pair. When we have found all five ordered pairs on the line, we can plot and join them to graph the line.

In fact, we only need two points. Can you justify why?

Carry out the plan

To find the missing coordinate, substitute the known part of each ordered pair into the equation of the line.

When $x = 1$, $\quad 1 + y = 4$, thus $y = 4 - 1 = 3$

The first point is (1, 3).

When $y = 2$, $\quad x + 2 = 4$, thus $x = 4 - 2 = 2$

The second point is (2, 2).

When $x = 3$, $\quad 3 + y = 4$, thus $y = 4 - 3 = 1$

The third point is (3, 1).

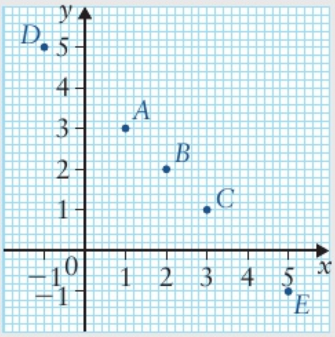
When $y = 5$, $x + 5 = 4$, thus $x = 4 - 5 = -1$

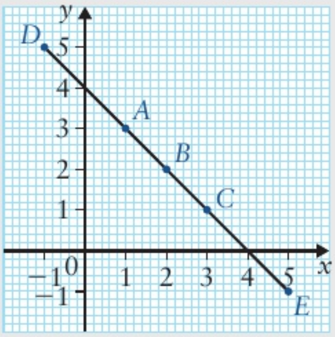
The fourth point is (−1, 5).

When $x = 5$, $5 + y = 4$, thus $y = 4 - 5 = -1$

The fifth point is (5, −1).

Now, plot these points $A(1, 3)$, $B(2, 2)$, $C(3, 1)$, $D(-1, 5)$ and $E(5, -1)$ on the coordinate plane and join them with a straight line. The line could be drawn by using any two of the points.





Look back

Is the solution true? Yes. When we add the x- and y-coordinates of the points A, B, C, D, E we get 4.

A: $1 + 3 = 4$, B: $2 + 2 = 4$, C: $3 + 1 = 4$, D: $-1 + 5 = 4$, E: $5 + (-1) = 4$

Worked example 5.3

A straight line has the equation $x - y = -3$

Identify 3 different points on the line. Plot and join the points to draw the graph of the line.

Solution

We are asked to find 3 different points on the line $x - y = -3$

The plan is to find 3 ordered pairs or points by using the equation of the line. We can select any 3 x-coordinates and find corresponding y-coordinates from the equation of the line. Then plot the points on the plane and connect two of them with a straight line.

To find the points, substitute selected x values into the equation and work out the corresponding y values.

When $x = 1$, $\quad 1 - y = -3, y = 4$

The first point is (1, 4).

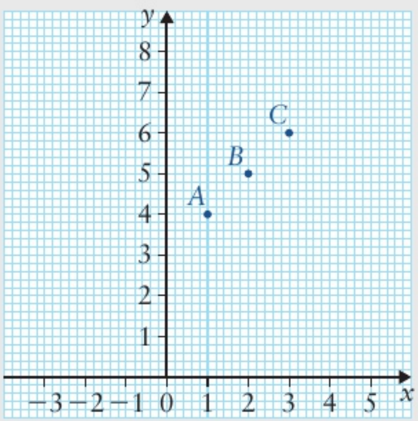
When $x = 2$, $\quad 2 - y = -3, y = 5$

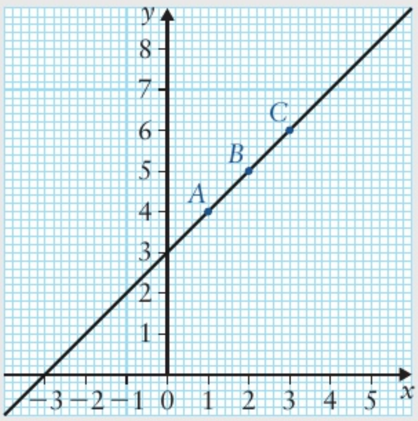
The second point is (2, 5).

When $x = 3$, $\quad 3 - y = -3, y = 6$

The third point on the line is (3, 6).

Plot the points $A(1, 4)$, $B(2, 5)$ and $C(3, 6)$ on the coordinate plane and join any two of them with a straight line.





Does the answer fit the equation? Yes. When we subtract the y-coordinates from the x-coordinates of the points A, B, C we get -3.
A: $1 - 4 = -3$, B: $2 - 5 = -3$, and C: $3 - 6 = -3$

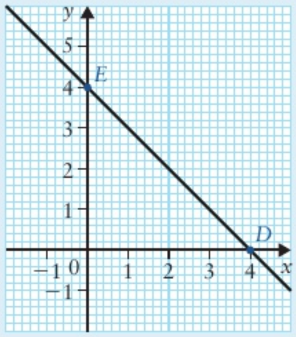
Reflect

Could you have drawn the lines in Worked examples 5.2 and 5.3 differently?

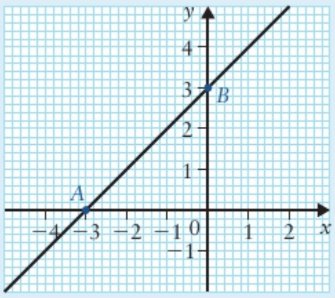
Hint

You only need to plot two points to draw a straight line. The x- and y-intercepts are easy to find because one of the coordinates is 0 at each of these points.

For Worked example 5.2, a simple way of drawing the line with equation $x + y = 4$ is to plot the x-intercept $(4, 0)$ and the y-intercept $(0, 4)$ and connect these points on the coordinate plane.



For Worked example 5.3, a simple way of drawing the line with equation $x - y = -3$ is to plot the x-intercept $(-3, 0)$ and the y-intercept $(0, 3)$ and connect these points on the coordinate plane.



Worked example 5.4

Does the point $(1, 3)$ lie on the straight line with equation $y = 3x + 2$?

Solution

We want to know whether the point $(1, 3)$ is on the straight line with equation $y = 3x + 2$

We can substitute the coordinates of the point $(1, 3)$ into the equation to see if it satisfies the rule.

Now, substituting $x = 1$ into $y = 3x + 2$ gives

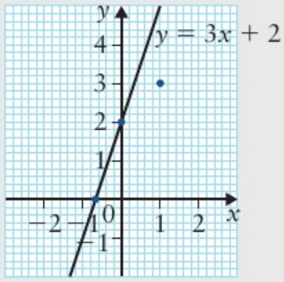
$$y = 3 \times 1 + 2$$
$$= 5$$

Hint

If a point is on a line, then the coordinates of the point must satisfy the equation of the line.

The y-coordinate of the point (1, 3) is 3. Since $3 \neq 5$, the point (1, 3) is not on the line $y = 3x + 2$

Looking back, the x- and y- intercepts of the line with equation $y = 3x + 2$ are $\left(-\frac{2}{3}, 0\right)$ and (0, 2) respectively. If we draw the line $y = 3x + 2$ and plot the point (1, 3) on a coordinate plane, we can see that this point is not on the line.



Practice questions 5.2

1 Copy and complete the table for each of these equations.

a $y = 2x$

x	0	1	2
y			

b $y = 3x + 1$

x	0	1	2
y			

c $y = 2x - 1$

x	0	1	2
y			

d $y = 2 - 3x$

x	0	1	2
y			

2 On separate coordinate grids, draw the graphs of each of the equations in question 1. What do you notice about the lines in parts a and c of question 1?

3 Find the x- and y-intercept of the graph of each of these equations.

a $y = 2x$

x	0	
y		0

b $y = 3x + 1$

x	0	
y		0

c $y = 2x - 1$

x	0	
y		0

d $y = 2 - 3x$

x	0	
y		0

4 For each equation in question 3, use the x- and y-coordinates only to sketch each graph.

5 Draw the graph of each of these equations by plotting the x- and y-intercepts only. After sketching all 4 lines, can you make an observation about the connection between the lines in parts a and c? What about those in parts b and d?

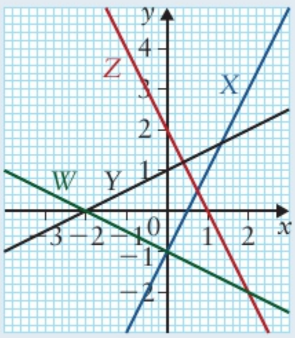
a $x + 2y = 4$ b $y = 2x - 1$

c $y = \frac{x}{2}$ d $2x + y - 4 = 0$

6 On which of these lines does the point $(1, -2)$ lie? Show how you work out your answer.

a $2x - 3y = 6$ b $x - 2y - 5 = 0$

7 Match each of these equations to its graph. Can you make an observation about the connection between the lines in parts a and c? What about those in parts b and d?



a $y = \frac{1}{2}x + 1$

b $y = 2x - 1$

c $y = -2x + 2$

d $y = -\frac{1}{2}x - 1$

8 Write the x-intercept and the y-intercept of each of the straight lines (a, b, c and d) in question 7.

9 Which of the points $(1, 4)$ and $(-2,-2)$ lies on the line $2x - y + 2 = 0$? Show how you work out your answer.

5.3 Horizontal and vertical lines

Explore 5.3

Plot the following points on a coordinate plane. Join them with a line. What do you notice?

Table 1

x	2	2	2	2
y	0	1	2	3

Table 2

x	0	1	2	3
y	3	3	3	3

How would you describe horizontal and vertical lines?

Consider the line through the points $(a, 0)$, $(a, 1)$, and $(a, 3)$, where a is any real number of your choice. Also, the line through $(0, b)$, $(1, b)$, and $(4, b)$. What do you notice?

Horizontal and vertical lines

Worked example 5.5

Draw the lines $x = 4$ and $y = -2$ on a coordinate plane. State whether the lines are horizontal or vertical.

Solution

We need to draw both the lines $x = 4$ and $y = -2$ on a coordinate plane by plotting points on the lines and connecting them.

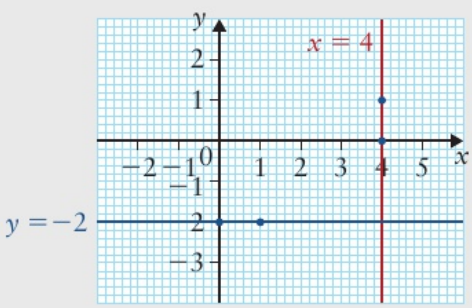
The plan is to plot a minimum of two points on each line and graph the lines by connecting these two points. We can use the definition of vertical and horizontal lines to identify them.

Now, $(4, 0)$ and $(4, 1)$ are two points on the line $x = 4$. $(0, -2)$ and $(1, -2)$ are two points on the line $y = -2$

The line $x = 4$ has the form $x = a$, so it is a vertical line.

The line $y = -2$ has the form $y = b$, so it is a horizontal line.

Here are the graphs of the lines.



Looking back, $x = 4$ is a vertical line; the x-coordinate of all the points on the line is 4 and the line cuts the x-axis at 4.

$y = -2$ is a horizontal line; all the points on the line have y-coordinate -2 and the line cuts the y-axis at -2.

Reflect

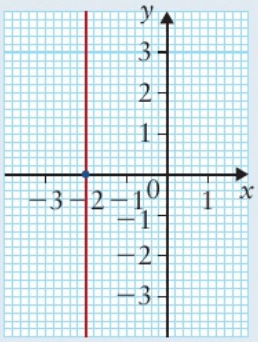
How do you use your GDC to graph $y = -2$ and $x = 4$?

How many points do you need to sketch a vertical or horizontal line?

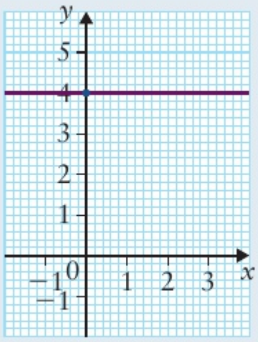
Practice questions 5.3

1 Write down the equation of each of these vertical or horizontal lines.

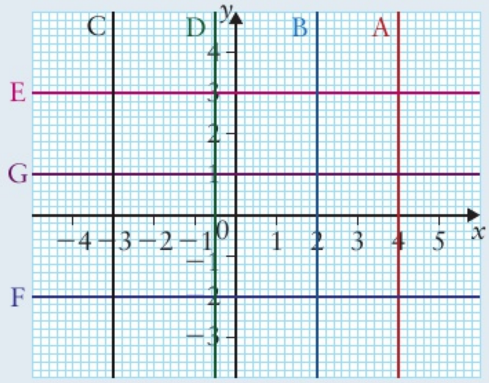
a



b



2 Write down the equation of each of the lines A to G.



3 Draw these vertical and horizontal lines on the same coordinate plane.

$x = 3$ $\quad x = -1$ $\quad y = -3$ $\quad y = -1$

What does the enclosed shape look like?

4 Write the coordinates of the points of intersection of all the lines A to G in question 2. How many intersection points did you count?

5 Does the point (4, 2) lie on the line $x = 4$? What about $y = 3$? Explain how you know.

6 Find the point of intersection of each pair of lines.

a $x = 2$ and $y = 5$
b $x = -3$ and $y = 5$
c $x = -4$ and $y = -3$
d $x = 0$ and $y = 2$
e $x = -5$ and $y = 0$
f $x = 0$ and $y = 0$

Challenge Q7

7 Find the point of intersection of each pair of lines.

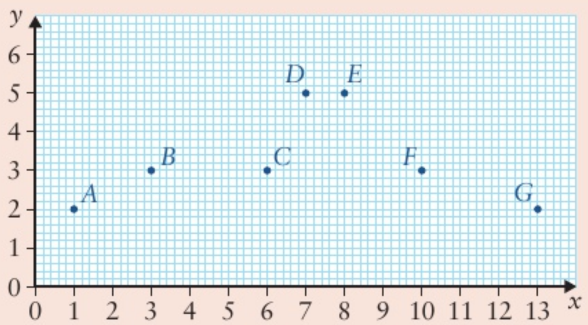
a $x + y + 1 = 0$ and $x = 2$
b $y = 3x - 4$ and $y = 2$
c $y = -\frac{2}{3}x$ and $x = 6$
d $2x - y + 3 = 0$ and $y = 1$
e $x = 0$ and $y = -x - 4$
f $y = 0$ and $y = \frac{1}{2}x - 3$

5.4 Gradient and equation of a line

5.4.1 Gradient

Investigation 5.1

The diagram shows part of a map. The letters represent towns. Jason and Kaan are travelling from A to G. Investigate the following questions.



Road sign showing a gradient

1 Find out which line segment out of each pair is the steepest.

a BC or DE
b AB or CD
c EF or FG

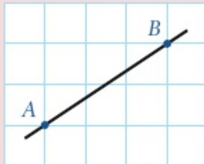
2 Identify if the path is sloping up, sloping down or not sloping for each of these routes.

a A to B
b B to C
c C to D
d D to E
e E to F
f F to G

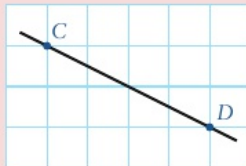
3 Describe a rule for giving the steepness of any part of a route (AB, BC, and so on). Justify your rule.

Fact

- If a line is horizontal, then there is no change in y, so we say that it has a zero gradient.
- If a line is vertical, then there is no change in x, so the gradient cannot be defined.
- A line sloping upwards from left to right is said to have a positive gradient. Line AB has a positive gradient.



- A line that slopes downwards from left to right is said to have a negative gradient. Line CD has a negative gradient.



Fact

The **gradient** or **slope** measures the steepness of a line.

Gradient can be defined as the ratio of rise to run.

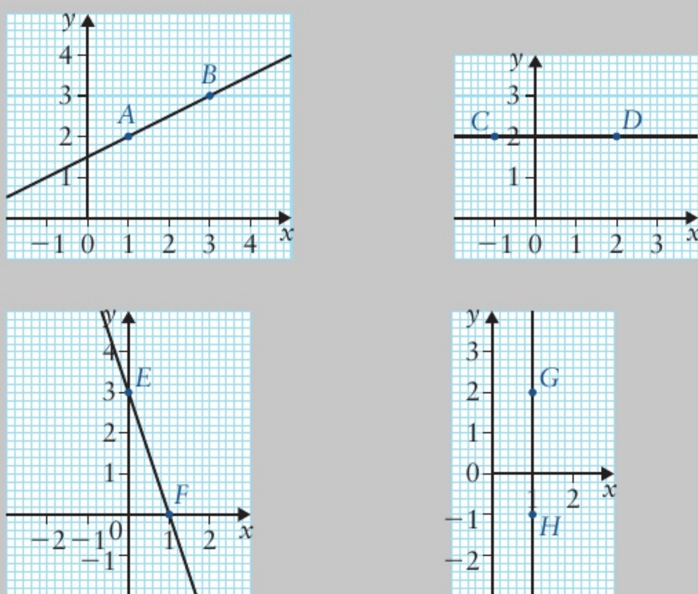
Gradient = $\dfrac{\text{rise (change in } y\text{)}}{\text{run (change in } x\text{)}}$

Fact

The gradient of a line is also called the **slope** of the line.

Worked example 5.6

Find the gradients of these lines.



Solution

We need to calculate the gradient of the lines AB, CD, EF and GH. We know the coordinates of two points on each line.

We use the formula for the gradient using the coordinates we know on each line. Apply the formula from left to right.

$$\text{gradient of } AB = \frac{\text{change in } y}{\text{change in } x} = \frac{1 \text{ up}}{2 \text{ right}} = \frac{1}{2}$$

$$\text{gradient of } CD = \frac{\text{change in } y}{\text{change in } x} = \frac{0}{3 \text{ right}} = 0$$

$$\text{gradient of } EF = \frac{\text{change in } y}{\text{change in } x} = \frac{3 \text{ down}}{1 \text{ right}} = \frac{-3}{1} = -3$$

$$\text{gradient of } GH = \frac{\text{change in } y}{\text{change in } x} = \frac{3 \text{ down}}{0} = \text{undefined}$$

Check that the calculated gradients make sense. Line AB has a positive gradient and the ratio of rise : run is 1 : 2. Line CD is a horizontal line so we know the gradient is zero. Line EF has a negative gradient and the ratio of rise : run is −3 : 1. Line GH is a vertical line and the gradient of a vertical line cannot be defined.

Architectural design often requires an understanding of gradients (slopes)

Reflect

Can you find another, more direct way of calculating the gradient?

Fact

- The gradient or slope of a line is usually denoted by a lower-case letter m.
- If the coordinates of two points on a line are $A(x_1, y_1)$ and $B(x_2, y_2)$, then the gradient of the straight line AB can be described as $m_{AB} = \frac{y_2 - y_1}{x_2 - x_1}$ or $m_{AB} = \frac{y_1 - y_2}{x_1 - x_2}$

5.4.2 Equation of a straight line: the gradient–intercept form

Hint

Go to

and click on the *start calculator* button. Then you can enter the equation $y = mx + c$ for each pair of m and c values and you will be able to draw the graphs.

Investigation 5.2

Use available software or a GDC for this investigation

1 On the same coordinate plane, sketch the lines with equation $y = mx + c$ for each pair of values for m and c given in Table 1.

Table 1

m	2	2	2	2
c	0	1	2	3

2 What do you notice about the graphs you drew in question 1?

3 Now use a new page to draw, on the same coordinate plane, the lines with equation $y = mx + c$ for each pair of values for m and c given in Table 2.

Table 2

m	0	1	2	3
c	1	1	1	1

4 What do you notice about the graphs you drew in question 3?

5 Sketch the straight lines given by each of the equations in Table 3. Identify the gradient and y-intercept for each line.

Table 3

Equation	Gradient	y-intercept
$y = x + 1$		
$y = 2x - 1$		
$y = x - 1$		
$y = 2x + 1$		
$y = \frac{1}{2}x - 2$		
$y = -3x + 4$		

6 Suggest a way to describe the relationship between the values of m and c and the graph of the equation $y = mx + c$. Justify your suggestion.

When the equation of a line is written in the form $y = mx + c$:

- m is the gradient
- c is the y-intercept.

$y = mx + c$ is called the **gradient–intercept** form of the equation of a straight line.

Fact

$y = mx + c$ is also called the **slope–intercept** form.

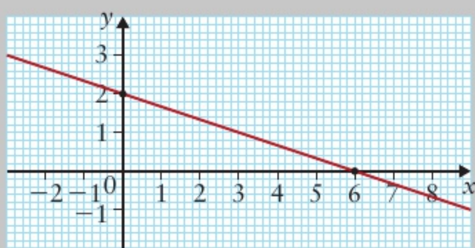
Worked example 5.7

a For the line given in the diagram, find:

i the gradient

ii the y-intercept.

b Write the equation of the line in gradient–intercept form.



Solution

We need to identify the gradient and y-intercept of the line. Note that the line has a negative gradient.

The plan is to use the gradient formula with any two points on the line to find the gradient, m. We can find the y-intercept by looking at where the graph cuts the y-axis.

a **i** Two points on the line are the y-intercept (where the graph cuts the y-axis) (0, 2) and the x-intercept (where the graph cuts the x-axis) (6, 0). We can substitute these values into the gradient formula to find m:

$$m = \frac{2-0}{0-6} = \frac{2}{-6} = -\frac{1}{3}$$

ii The y-intercept has coordinates (0, 2).

b In $y = mx + c$, m is $-\frac{1}{3}$ and c is the y-coordinate of the y-intercept, which is 2.

So the equation of the line is $y = -\frac{1}{3}x + 2$

To check our solution, we can find the x-intercept by substituting in $y = 0$:

$$0 = -\frac{1}{3}x + 2, \frac{1}{3}x = 2, x = 6$$

This gives (6, 0) as the x-intercept, as required.

The y-intercept is where $x = 0$:

$$y = -\frac{1}{3} \times 0 + 2 = 2$$

This gives (0, 2) as the y-intercept. So both points satisfy the given graph.

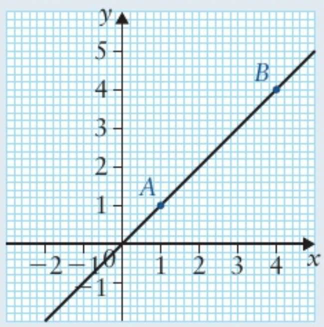
Reflect

Can the equation be found by a different method?

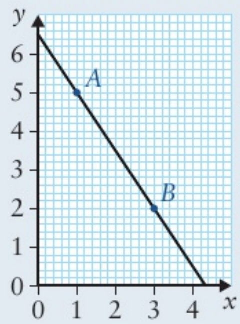
Practice questions 5.4

1 Find the gradient of each line AB and then write its equation.

a



b

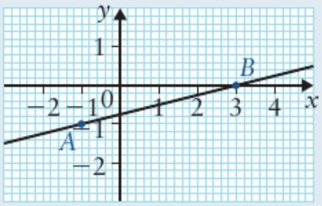


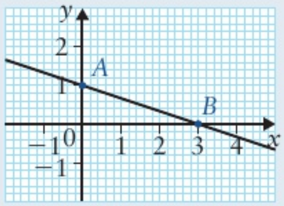
Fact

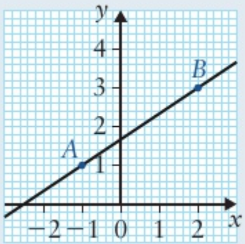
The equation of a straight line can be represented in different forms:

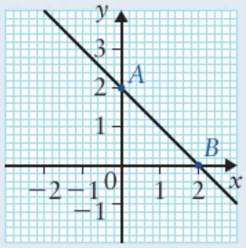
- gradient–intercept form: for example, $y = 2x + 3$
- general form: for example, $2x - y + 3 = 0$ or $y - 2x = 3$

Both of these forms represent the same graph.

c


d


e


f


Connections

See Chapter 6 for how to write an equation in different forms.

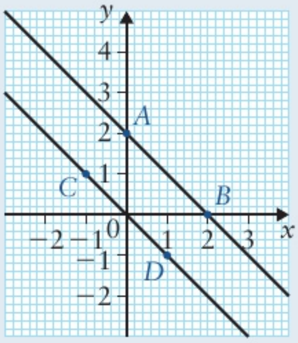
2 Find the gradient of the line passing through each pair of given points and then write its equation.

a $A(1, 1)$ and $B(2, 3)$

b $C(-1, 0)$ and $D(0, -1)$

c $E(3, 1)$ and $F(2, 4)$

d $G(0, 1.5)$ and $I(-1.5, 3)$

3


Find the gradients of the lines AB and CD.
Are the lines parallel?

Hint Q3

Parallel lines have the same gradient.

4 Identify the gradient, m, and the y-intercept, c, of each line.

a $y = 2x + 5$

b $y = x - 1$

c $y = -x + 3$

d $y = 5x - 1$

e $y = 3x$

f $x = 3$

g $y = 4$

h $3x - 2y = 4$

i $x - y + 1 = 0$

j $\frac{-1}{2}x - \frac{2}{3}y = 2$

Hint Q4

Use the gradient–intercept form $y = mx + c$

Challenge Q4g

5 Draw the graph of each of these straight lines, given the gradient, m, and y-intercept, c.

a Gradient is -2 and y-intercept is 2

b Gradient is 3 and y-intercept is -2

c $m = 2$ and $c = 0$

d $m = -1$ and $c = 5$

e Gradient is $\frac{1}{2}$ and y-intercept is 1

6 Lines are parallel if they have the same gradient. Which of these pairs of lines are not parallel?

a $y = 2x - 5$ and $y = 2x + 1$

b $y = -x + 3$ and $y = 1 - x$

c $y = 3x - 2$ and $y = 2x - 3$

Challenge Q6d

d $x + y = 2$ and $-2x - 2y - 4 = 0$

7 Match each equation with the correct line.

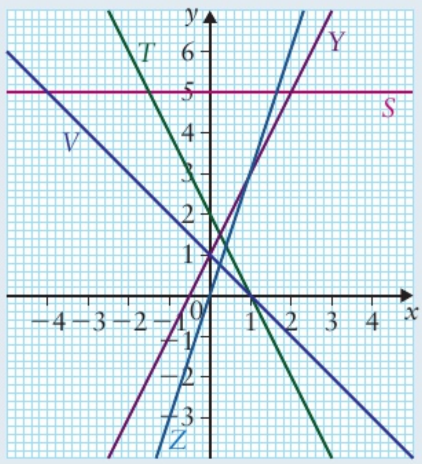
a $y = 2x + 1$

b $y = 3x$

c $y = -x + 1$

d $y = 5$

e $y = 2 - 2x$



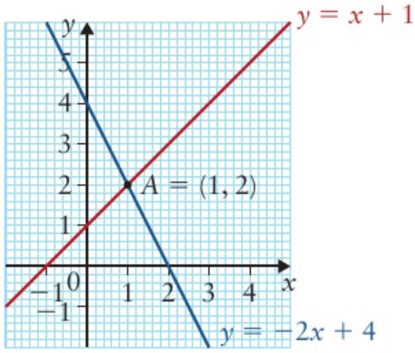
5.5 Intersection of two lines

Explore 5.4

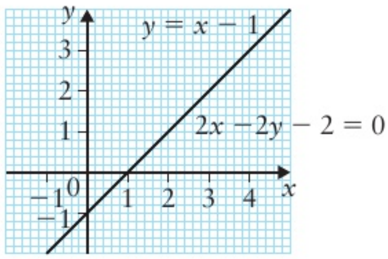
Without graphing two lines, how can you tell whether they intersect?

At how many different points can two lines intersect?

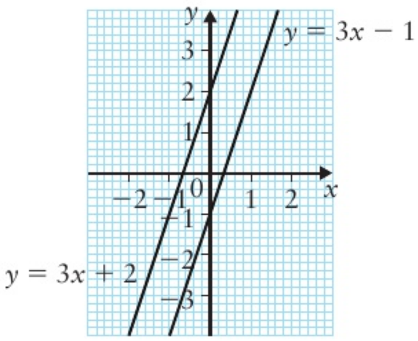
- Two lines intersect if we can find a point that satisfies both of their equations. For example, the lines $y = -2x + 4$ and $y = x + 1$ intersect at $A(1, 2)$.



- If two lines intersect at more than one point, then they are the same line. For example, $y = x - 1$ and $2x - 2y - 2 = 0$ are the same line and they intersect at every point.



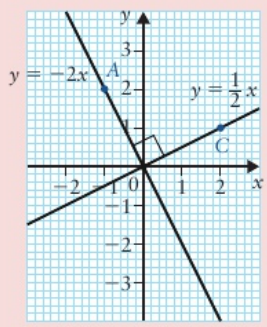
- If two lines do not intersect at any point, then they are parallel lines.



For example, $y = 3x - 1$ and $y = 3x + 2$ have no points of intersection because they are parallel.

Fact

If two lines are perpendicular (they cross at right angles), then the product of their gradients is -1.



Product of the gradients of the lines is $\frac{1}{2} \times -2 = -1$

Worked example 5.8

Find the point(s) of intersection of each pair of lines.

a $y = x + 1$ and $y = 1 - x$

b $y = 3$ and $y = 2x + 1$

c $x + y = 2$ and $y = x$

Solution

For each pair of lines, we need to work out if there are any intersection points. We know how to draw the graph of a straight line from its equation.
To find the points of intersection, we can graph the lines using their equations and see if they intersect.

Intersecting airport runways

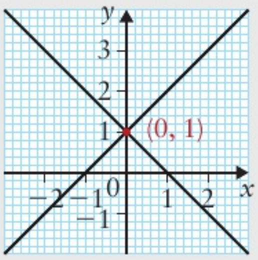
a The tables give three points on each line.

$y = x + 1$

x	−1	**0**	1
y	0	**1**	2

$y = 1 - x$

x	−1	**0**	1
y	2	**1**	0



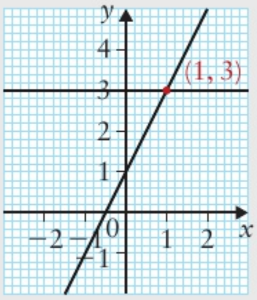
We can see from the table that (0, 1) is the point of intersection of the lines. The graphs of the lines $y = x + 1$ and $y = 1 - x$ are shown in the diagram.

b $y = 3$

x	−1	0	**1**
y	3	3	**3**

$y = 2x + 1$

x	−1	0	**1**
y	−1	1	**3**



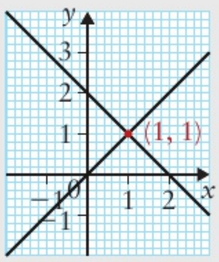
The tables show that (1, 3) is the point of intersection.
The graphs $y = 3$ and $y = 2x + 1$ are shown in the diagram.

c $x + y = 2$

x	-1	0	**1**
y	3	2	**1**

$y = x$

x	-1	0	**1**
y	-1	0	**1**



The tables show that (1, 1) is the point of intersection.
The graphs of $x + y = 2$ and $y = x$ are shown in the diagram.

The last step is to check whether our solutions make sense.
We can substitute the coordinates of each point of intersection into the equations of each pair of lines to see if the solutions are correct.

For $y = x + 1$ and $y = 1 - x$, substitute (0, 1):

$y = x + 1$, $1 = 0 + 1$, $1 = 1$ (true)

$y = 1 - x$, $1 = 1 - 0$, $1 = 1$ (true)

For $y = 3$ and $y = 2x + 1$, substitute (1, 3):

$y = 3$, $3 = 3$ (true)

$y = 2x + 1$, $3 = 2 \times 1 + 1$, $3 = 3$ (true)

For $x + y = 2$ and $y = x$, substitute (1, 1):

$x + y = 2$, $1 + 1 = 2$, $2 = 2$ (true)

$y = x$, $1 = 1$ (true)

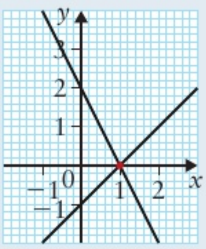
Connections

We can also find the points of intersection of two lines algebraically. We call this solving a system of equations or solving simultaneous equations.

Practice questions 5.5

1 Find the point of intersection of each pair of straight-line graphs.

a



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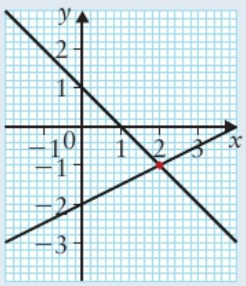

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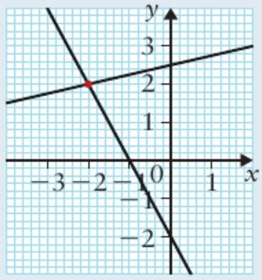
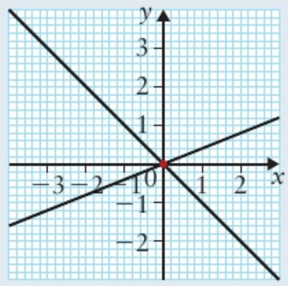


d



Hint Q2

You can find the intersection between lines using your GDC.

2 Find the intersection points of each pair of lines.

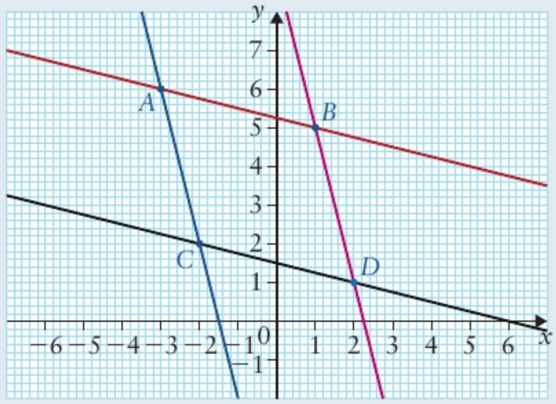
a $x + y = -2$ and $y = x$

b $2x - 3y - 6 = 0$ and $y = \frac{-1}{3}x + 1$

c $y = -x + 2$ and $x - 2y - 2 = 0$

d $y = 2x + 2$ and $y = 2x - 1$

3 The graphs of four straight lines are shown in the diagram.



a Find the point of intersection of each pair of lines.

i AB and AC ii AB and BD

iii CD and AC iv CD and BD

Challenge Q3b

Challenge Q3c

b What geometrical shape is $ABCD$? How do you know?

c Will AC and BD ever meet? Explain your answer.

4 Show that the lines $2x - 3y + 9 = 0$ and $-2x - 3y - 9 = 0$ intersect at $(-4.5, 0)$.

5 Find the points of intersection of the line $2x + y = 6$ with the x-axis and the y-axis.

6 Show that $x - 2y = 1$ and $y = 2x + 1$ are perpendicular lines.

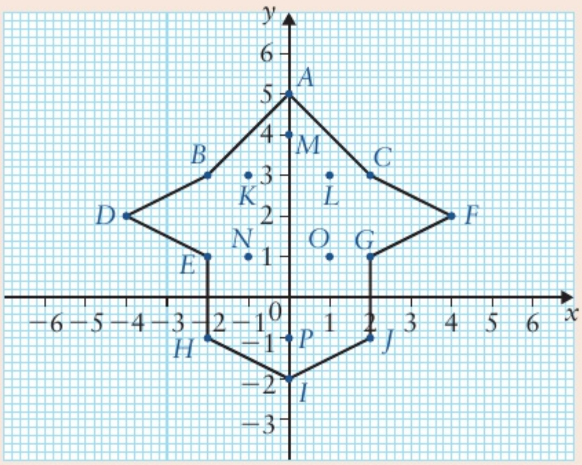
7 Show that $2x - 4y + 6 = 0$ and $y = \frac{1}{2}x - 1$ are parallel lines.

Self assessment

- I can identify ordered pairs on a coordinate plane.
- I can graph points on a coordinate plane.
- I can find the distance between two points.
- I can draw a straight-line graph on a coordinate plane.
- I can find the x-intercept of a straight-line graph.
- I can find the y-intercept of a straight-line graph.
- I can determine whether or not a point lies on a straight line.
- I can use the gradient formula.
- I can use a GDC or available software to draw straight lines.
- I can write the equation of a straight line in the form $y = mx + c$
- I can explain what is meant by the gradient–intercept form of the equation of a straight line.
- I can use different forms of the equation of a straight line to draw its graph.
- I can identify a horizontal line.
- I can identify a vertical line.
- I can represent the x-axis and y-axis as vertical and horizontal lines.
- I can find the gradient of a straight line.
- I can identify positive and negative gradients and describe them.
- I can explain the steepness of a straight line.
- I know that gradient and slope are the same thing.
- I can explain whether or not two lines are parallel.
- I can explain whether or not two lines are perpendicular.
- I can find the points of intersection of two lines geometrically.
- I can determine whether or not two lines will intersect.

Check your knowledge questions

Questions 1–3 refer to this diagram.



1 Name the points whose coordinates are given by each of these ordered pairs.

a (0, 4) **b** (−2, 1) **c** (2, −1) **d** (2, 3)

2 Write down the coordinates of each of these points.

a N b H c F d B

3 Find the distance between each pair of points.

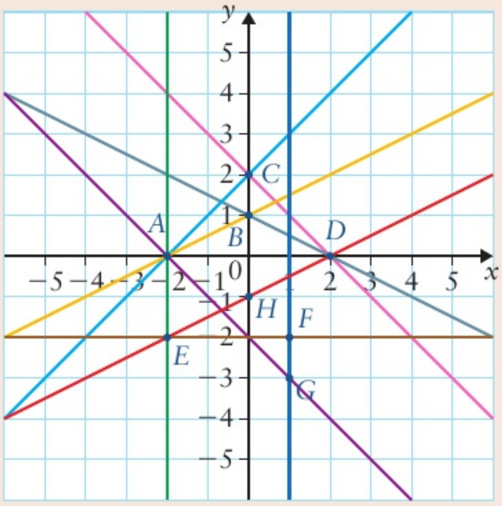
a M and P
b D and F
c H and J
d B and H
e A and F

Challenge Q3e

4 Graph the straight line given by each of these equations.

a $y = 3x - 1$
b $y = -2x + 1$
c $y = 2x - 2$
d $x + y = 3$

Questions 5–8 refer to this diagram.



5 Find the equation of each line.

a AE b AD c BC
d EF e FG

6 Find the gradient of each line.

a AB b AC c CD
d BD e FG f EF

7 Find the equation of each line in gradient–intercept form.

a AG b AB c BD

d CD e ED

8 Find the coordinates of the point of intersection of each pair of lines.

a AG and EF b AB and BD c AD and BC

d AB and ED e AG and CD

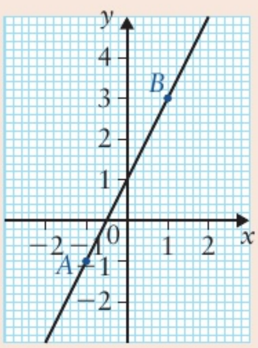
9 State the x-intercept and the y-intercept of each of these lines.

a $y = 3x - 5$ b $y = -x + 1$

c $x - y - 3 = 2$ d $2x + 3y = 12$

10 Find the equation of the straight line passing through points $A(1, 2)$ and $B(0, 3)$.

11 The graph shows the line AB.

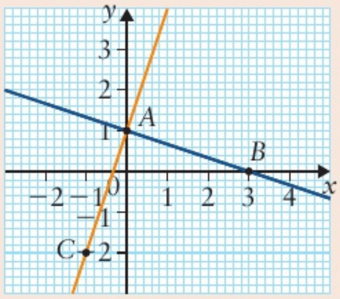


Find:

a the x-intercept b the y-intercept

c the equation of the line in gradient–intercept form.

12 The diagram shows the lines AB and AC.



a Find:

i the equation of the line AB

ii the equation of the line AC.

Challenge Q12b

b What do you observe about the lines AB and AC?

13 Determine whether each point lies on the given line.

a $A(1, 2)$ and $y = 2x - 1$

b $B(-1, 1)$ and $x + y = 0$

c $C(2, -1)$ and $x - 2y + 5 = 0$

d $D(0, 1)$ and $2x - 5y = -5$

6 Equations and inequalities

6 Equations and inequalities

KEY CONCEPT

Logic

RELATED CONCEPTS

Change, Equivalence, Representation, Systems

GLOBAL CONTEXT

Identities and relationships

Statement of inquiry

Representing changing relationships between variables with a logical system allows us to solve real-life problems.

Factual

- How are equivalent equations created?
- How are inverse operations used?

Conceptual

- How are equations and inequalities used to model real-world situations?

Debatable

- Can two measurable quantities really be equal?

Do you recall?

1 Find the solution to $a + 3 = 10$

2 Is 3 a solution to the equation $4 - n = -1$?

3 Simplify the following:

a $\frac{7}{36} + \frac{5}{54}$ b $7 + 3x^2 + 5x^2y - 2x^2 - 3xy^2 + x^2 - 2xy^2$

c $12 - 2[(3-6)^2 + 1] \div 4$

6.1 Solving simple equations

Previously, we solved equations by inspection or by performing one inverse operation. In this section, we will see how to solve equations involving more than one step.

We will combine like terms in order to reduce equations to simple equivalent equations.

Explore 6.1

Can you write these expressions in simpler forms?

a $6 - 4(2t - 5 + 3t) + t - 9$

b $-3(4 - 3a) + 5(a + 2) - (10 + 7a - 12)$

Did you use more than one way to perform the task?

Worked example 6.1

Solve $4g - 1 = -9$ for g.

Solution

In order to solve the equation for g, use inverse operations to isolate g on one side of the equation, and then solve to find its value.

$$\begin{array}{r} 4g - 1 = -9 \\ +1 \quad +1 \\ \hline 4g = -8 \end{array}$$

First, add 1 to **both** sides.

$$\frac{4g}{4} = \frac{-8}{4}$$

Then, divide both sides by 4 to isolate the variable.

$$g = -2$$

The solution is $g = -2$

To check the solution, substitute the value of g into the original equation and simplify.

$4(-2) - 1 = -9$	Substitute the value of g, and follow the order of operations to simplify.
$-8 - 1 = -9$	Work from left to right.
$-9 = -9$	Because the left-hand side equals the right-hand side, the solution is correct.

When solving an equation using inverse operations, we must keep the equation balanced. Whatever operation we perform on one side of the equation, we must perform on the other side as well.

Worked example 6.2

Solve $-5 = 12y - 3 - 8y$ for y.

Solution

In order to solve the equation for y, use inverse operations to isolate y on one side of the equation, and then solve.

$-5 = 4y - 3$	Combine like terms on the right-hand side to simplify.
$+3 \qquad +3$	Next, add 3 to both sides.
$-2 = 4y$	
$\frac{-2}{4} = \frac{4y}{4}$	Divide both sides by 4 to isolate the variable.
$-\frac{1}{2} = y$	The solution is $y = -\frac{1}{2}$

To check the solution, substitute the value of y into the original equation and simplify.

$-5 = 12\left(-\frac{1}{2}\right) - 3 - 8\left(-\frac{1}{2}\right)$	Substitute the value of y and follow the order of operations to simplify.
$-5 = -6 - 3 + 4$	Work from left to right; subtract, then add.
$-5 = -5$	Because the left-hand side equals the right-hand side, the solution is correct.

Worked example 6.3

Your class wants to raise €2000 for a local animal sanctuary. The plan is to make and sell gift boxes in small and large sizes. They will charge €5 for small boxes and €9 for large boxes. The class has a budget of €450 to spend on materials to make the boxes. The class wants to make the same number of small and large boxes.

How many of each size box need to be made?

Solution

Understand the problem

- The question is asking about the number of boxes that need to be made and sold in order to raise €2000.
- Take into account the €450 cost of the materials.
- The class needs to make a profit of $2000, so the money earned from selling the boxes (the revenue) must be equal to the money spent on materials plus €2000.
- We can use a basic business model: profit = revenue − cost of materials

Make a plan

- Let b = the number of boxes of each type to be made.
  Create an equation following this business model:
  profit = revenue − cost of materials
- The **revenue** from selling the small and large boxes is $5b + 9b$
- The **cost of materials** is €450.
- The **profit** is the amount of money needed at the end: €2000.

Carry out the plan

revenue − cost = profit

$5b + 9b - 450 = 2000$

$14b - 450 = 2000$	Combine like terms on the left-hand side to simplify.
$+450 \quad +450$	Add 450 to both sides.
$14b = 2450$	
$\frac{14b}{14} = \frac{2450}{14}$	Divide both sides by 14 to isolate the variable.
$b = 175$	The solution is $b = 175$

Therefore, the class will need to make 175 small boxes and 175 large boxes to reach their target of raising €2000.

Look back

Check the solution by substituting 175 into the original equation and simplifying:

$$5(175) + 9(175) - 450 = 2000$$
$$2450 - 450 = 2000$$
$$2000 = 2000$$

Connections

Mathematics can help you to solve problems in business.

Reflect

Look back at the questions from Explore 6.1.

Can you solve the worked examples using different steps?

Suppose we need to produce different numbers of small and large boxes. Would the method we used so far work?

Thinking skills

Practice questions 6.1

Solve these equations. Remember to check your solutions.

Reminder

Don't forget to look for like terms that can be combined, to create a simpler equation.

1 $2a + 5 = 7$

2 $10 = 4p - 6$

3 $18 - 7y = 4$

4 $19 = -7 - 13s$

5 $-4n - 9 = 11$

6 $-6 + 5d = 9$

7 $-17 = -5 + 16m$

8 $9k + 6 = -12$

9 $11c - 8 - 6c = 27$

10 $42 = 11 - 5v + 7$

11 $4u + 8u + 14 = -10$

12 $23 + 6y - 18 = -13$

13 $-3 = 6j - 14j + 2$

14 $-14 - 2z + 6 = 0$

For qestions 15–24, work out a solution to each situation.

Reminder

Remember to include Pólya's steps in your solution.

15 A rectangle is 5 times as long as it is wide. The perimeter is 1200 cm. What are the dimensions of the rectangle?

16 Alex has 68 pencils. He gave 3 pencils to each of his friends, and he still had 8 pencils for himself. How many friends does Alex have?

17 Jiwon, Paul and Dominic all study mathematics. To complete a homework assignment, Paul takes twice as many minutes as Jiwon, and Dominic takes twice as many minutes as Paul. All together, they take 112 minutes. How many minutes does each student spend on the homework assignment?

18 Jad has £21. After he buys 4 boxes of cereal, he has £13. How much does each box of cereal cost?

19 Sebastian has some euros. Floriane has 3 more than twice as many euros as Sebastian. Together they have €54. How many euros do they each have?

20 A book of poems has 100 fewer pages than a book on history. Together the books have 250 pages. How many pages does each book have?

21 Write this statement as an equation and then find the number: sixteen less than five times a number is 114.

22 The cost for work done by an electrician was €348. The electrician charges a flat fee of €95 and has an hourly rate of €46. For how many hours did the electrician work?

23 The sum of three consecutive odd integers is −147. What are the three integers?

24 Five swimming pools need to be filled. Each pool holds 1 250 000 litres of water. Two pumps are used to fill the pools. The first pump is able to pump water at a rate of 34 000 litres per hour, and the second pump has a rate of 46 000 litres per hour. Used together, how long will it take for the two pumps to fill all five pools?

Challenge Q24

6.2 Solving equations with variables on both sides

In 2020, a city's population was 9.9 million and was increasing by 0.6 million each year. The population of the city suburbs was 11.4 million, increasing by 0.5 million each year. When will the two have the same population?

To find a solution to such a situation, you need to set up an equation with variables on both sides. There are many such situations in real life, and this section will discuss methods of solving such equations.

Explore 6.2

Can you set up an equation to describe the situation above?

Attempt to find a solution.

Worked example 6.4

Solve $12d + 7 = 6d - 11$ for d.

Solution

In order to solve the equation for d, use inverse operations to isolate d on one side of the equation, and then solve.

$$\begin{array}{rcl} 12d + 7 &=& 6d - 11 \\ -7 & & -7 \\ \hline \end{array}$$

Subtract 7 from both sides.

$$\begin{array}{rcl} 12d &=& 6d - 18 \\ -6d & & -6d \\ \hline \end{array}$$

Subtract $6d$ from both sides.

Reminder

We want to isolate the variable, so move terms with a variable to one side of the equals sign, and constants (terms without a variable) to the other side.

$\frac{6d}{6} = \frac{-18}{6}$	Divide both sides by 6 to isolate the variable.
$d = -3$	The solution is $d = -3$

To check the solution, substitute your value of d into the original equation and simplify.

$12(-3) + 7 = 6(-3) - 11$	Substitute the value of d into the equation, and follow the order of operations to simplify it.
$-29 = -29$	The left-hand side equals the right-hand side, so the solution is correct.

Hint

To **solve** an equation **with variables on both sides**, simplify one or both sides of the equation, if necessary. Then use inverse operations to collect the variable terms on one side, collect the constant terms on the other side, and isolate the variable.

Worked example 6.5

Solve $3x + 17 - 5x = -4 + 2x - 7$ for x.

Solution

In order to solve the equation for x, use inverse operations to isolate x on one side of the equation, and then solve for its value. We can start by identifying and combining like terms.

$-2x + 17 = -11 + 2x$	Combine like terms on both sides.
$+2x \qquad +2x$	Add $2x$ to both sides.
$17 = 4x - 11$ $+11 \qquad +11$	Add 11 to both sides.
$\frac{28}{4} = \frac{4x}{4}$	Divide both sides by 4 to isolate the variable.
$7 = x$	The solution is $x = 7$

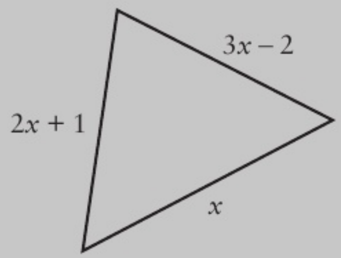
To check the solution:

$3(7) + 17 - 5(7) = -4 + 2(7) - 7$	Substitute the value of x into the equation, and follow the order of operations to simplify it.
$38 - 35 = 10 - 7$	Simplify.
$3 = 3$	The left-hand side equals the right-hand side, so the solution is correct.

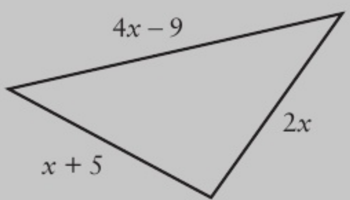
Worked example 6.6

These two triangles have the same perimeter. What is the perimeter of each triangle?

Triangle 1



Triangle 2



Understand the problem

This question is linked to geometry and is about perimeter. Recall that the perimeter of a shape is found by adding all the sides together.

Make a plan

To create an equation, we first need to create an expression for the perimeter of each triangle. We then set these expressions equal to each other to form an equation that we can solve for the variable x. Once we know the value of x, we can substitute the value into one of the expressions to find the perimeter of each shape.

Connections

Here we see how algebra helps with geometry.

Carry out the plan

Expression for the perimeter of triangle 1: $2x + 1 + x + 3x - 2$

Expression for the perimeter of triangle 2: $x + 5 + 4x - 9 + 2x$

Set the two expressions equal to each other and solve for x.

$$2x + 1 + x + 3x - 2 = x + 5 + 4x - 9 + 2x$$

$6x + 1 - 2 = 7x + 5 - 9$	Combine like terms on each side to simplify the terms in x.
$6x - 1 = 7x - 4$	Then simplify the constant terms.
$-6x \qquad -6x$	Subtract $6x$ from both sides.
$-1 = x - 4$	Add 4 to both sides.
$+4 \qquad +4$	
$3 = x$	The solution is $x = 3$

Looking back, we can do this by checking that we solved the equation correctly.

$2(3) + 1 + (3) + 3(3) - 2 = (3) + 5 + 4(3) - 9 + 2(3)$	Substitute the solution into the equation, and follow the order of operations to simplify it.
$19 - 2 = 20 - 9 + 6$	Then combine the terms on each side, going from left to right.
$17 = 17$	The left-hand side equals the right-hand side, so the solution is correct.

While checking whether the solution makes sense, we discovered that the perimeter of each triangle is 17 units.

Reflect

Can you check your answer to Worked Example 6.6 in a different way?

Practice questions 6.2

Solve these equations.

1 $3a - 10 = 5a + 4$

2 $9 - 4r = 6r - 11$

3 $8t - 13 = 21 + 6t$

4 $-7n + 18 = 11n + 9$

5 $18 + 4p = -8p + 15$

6 $3l + 11l + 7 = 8l - 11$

7 $6k - 19 - 12k = 12k - 7$

8 $29 + 9w - 7 = 7w - 4$

9 $-12q + 17 = 6 - 9q - 10$

10 $42 + 13z = -7z + 15 - 33$

11 $-g + 37 = 4g + 17 + 3g$

12 $-5y + 15 + 11y = -6 + 2y - 19$

13 $8b + 27 - 15b = 8 - 3b + 19$

14 $2v - 9 + 5v = 22 - 2v - 7$

15 $-12 + 7y - 15 = -5y + 9 + 3y$

16 $20x - 17 + 8x - 25 = 13 + 14x + 17 + 5x$

17 $10 + 9d - 8 + 6d = 7d - 1 - 4d - 13$

18 $7t - 28 - 9t - 17 = 9 - 5t + 6 + 8t$

For questions 19–28, work out a solution to each situation.

19 Translate this statement into an equation and determine the value of the variable:

Seven more than six times a number is equal to eleven less than three times the same number. What is the number?

20 Hanane is looking to join a fitness club where she can also attend classes. BFit has a one-time membership fee of \$135 and charges \$5 per class. Fitness4U charges \$8 per class and has a one-time membership fee of \$90.

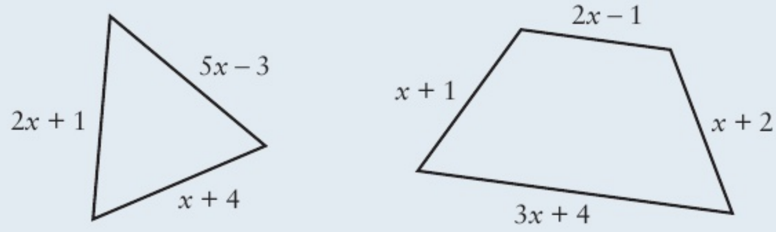
a If Hanane attends 10 classes per month, which club is cheaper?

b If Hanane attends 20 classes per month, which club is cheaper?

c Write and solve an equation to help Hanane determine which club she should join.

21 Currently, Ambia has \$85 and her brother Samir has \$160. Ambia is going to save \$7 each week, and Samir is going to spend \$8 each week. How many weeks will it be before Ambia and Samir have the same amount of money?

22 These two shapes have the same perimeter. Find the side lengths of each shape.



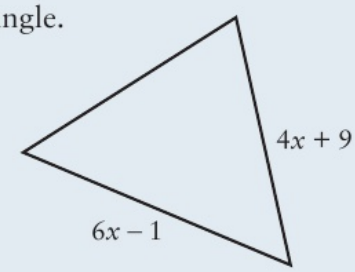
Hint Q23

Sketch the shape and write expressions for the length and width.

23 A rectangle's length is 6 units more than its width. The perimeter of the rectangle is 10 times its width.

Find the length and width of the rectangle.

24 All the sides of this triangle are the same length.
Find the perimeter of the triangle.



25 Orhan needs to hire a plumber. One plumber charges a fee of £75 plus an additional £27.50 per hour worked. Another plumber charges a fee of £120 plus an additional £23 per hour worked. For how many hours' work will the cost be the same for both plumbers?

26 Translate this statement into an equation and find the value of the variable:

Thirteen less than twice a number is equal to fourteen more than five times the same number. What is the number?

27 Sarah has €400 and earns €30 per day. Jonathan has €500 and earns €20 per day. How many days will it be before they have the same amount of money?

28 Selma sells 12 ears of corn for \$6. It costs her \$4 per 12 ears to pick the corn and \$50 per day to run the shop where she sells them. How many ears must she sell each day in order to break even?

6.3 Solving equations with brackets

In this section, we will explore how brackets (grouping symbols) are used when solving equations. This uses the distributive property of brackets.

When using the distributive property, it is very important to be careful with the sign (+) or (−) when multiplying.

Recall:

- When multiplying two positive numbers or two negative numbers, the result is a positive number.
- When multiplying a positive number and a negative number, the result is a negative number.

Explore 6.3

Simplify:

$-14 + 2(3 - 7)^2 + 5$

Can you identify the error a student is likely to make when doing this task?

Reflect

Why is it important to understand and follow a set of rules, such as the order of operations?

Worked example 6.7

Solve this equation for a.

$-2(6a + 1) = -4(a - 4)$

Solution

In order to solve the equation for a, use inverse operations to isolate a on one side of the equation, and then solve for its value. We can start by using the distributive property and then identify and combine any like terms.

$-2(6a + 1) = -4(a - 4)$ $-12a - 2 = -4a + 16$	Start to simplify the equation by expanding the brackets on both sides.
$+4a \qquad\qquad +4a$	Add $4a$ to both sides.
$-8a - 2 = 16$ $+2 \quad +2$	Add 2 to both sides.
$\frac{-8a}{-8} = \frac{18}{-8}$	Divide both sides by −8 to isolate the variable.
$a = -\frac{9}{4}$	The solution is $a = -\frac{9}{4}$

To check the solution, substitute the value of a into the original equation and simplify.

$-2\left(6\left(-\frac{9}{4}\right)+1\right)=-4\left(\left(-\frac{9}{4}\right)-4\right)$	Substitute the solution into the equation, and follow the order of operations to simplify it.
$-2\left(-\frac{54}{4}+\frac{4}{4}\right)=-4\left(-\frac{25}{4}\right)$	Remember: to combine fractions, they need to have a common denominator.
$25=25$	The left-hand side equals the right-hand side, so the solution is correct.

Worked example 6.8

A school has been collecting information about the number of students enrolled in different performing arts classes. Currently, there are 410 students studying music. An average of 22 additional students enrol for the music class each year. There are 165 students currently studying drama. On average, 3 students leave the drama course each year. If these trends continue, how long will it be before there are four times as many students studying music as drama?

Solution

This question is asking us to compare two quantities and determine when one will be four times as large as the other. In this case, there are more students taking music each year and fewer students taking drama each year.

We will need to write expressions for the number of students taking music and the number of students taking drama. Then form and solve an equation to determine when the number of music students will be four times that of the drama students. Finally, check the solution and answer the question.

Let t represent the number of years.

Expression for number of music students: $410 + 22t$

Expression for number of drama students: $165 - 3t$

We want to find out when there will be four times as many students taking music as there are taking drama, which means we need to multiply the expression for drama by 4 and set that equal to the expression for students studying music.

Then we can write the equation and solve for t.

$410 + 22t = 4(165 - 3t)$	Start to simplify the equation by expanding the brackets.
$+12t \qquad +12t$	Add $12t$ to both sides.
$410 + 34t = 660$ $-410 \qquad -410$	Subtract 410 from both sides.
$\dfrac{34t}{34} = \dfrac{250}{34}$	Divide both sides by 34 to isolate the variable.
$t \approx 7.3529\ldots$	The solution is $t \approx 7.35$

Make sure that the solution is correct and interpret it in order to answer the question. We can check the answer by substitution.

$410 + 22\left(\dfrac{250}{34}\right) = 4\left(165 - 3\left(\dfrac{250}{34}\right)\right)$	Substitute the solution into the equation, and follow the order of operations to simplify. It is better to use the exact value of $\dfrac{250}{34}$ rather than 7.35, which was rounded to 2 decimal places.
$410 + \dfrac{2750}{17} = 4\left(165 - \dfrac{375}{17}\right)$	First multiply.
$\dfrac{9720}{17} = \dfrac{9720}{17}$	Then combine the terms on each side, going from left to right. The left-hand side equals the right-hand side, so the solution is correct.

In approximately 7.35 years, there will be four times as many students studying music as studying drama.

Looking back, when we checked the solution for this problem, we used exact values. Does this make sense in the context of the problem? Would we get the same result using a rounded value? To what level of accuracy should we round the answer?

Reflect

- Thinking skills
- Social skills
- Communication skills

We have now seen several different types of equations and different ways of solving them. Look back at Explore 6.3. Working in a group, determine a set of general guidelines to follow for correctly solving each type of equation. Is it possible to determine one set of rules that always works or are there multiple ways to approach solving problems of this kind?

Practice questions 6.3

Solve these equations.

1 $3(2t - 5) = 21$

2 $-2(8 - 5a) = 14$

3 $-4(3k + 7) = 20$

4 $-5(5 - 2a) = 3a + 10$

5 $-3(8h - 5) = -9h - 10$

6 $9(2m - 8) = 13m - 32$

7 $3(2d - 4) = -2(d - 8)$

8 $-3(4 - 6n) = -4(-n - 4)$

9 $2(5q - 2) = -5(2q + 8)$

10 $8(7 - 3f) = 5(4 - 5f)$

11 $4(3 - h) = -7(2h + 4)$

12 $3(3g - 7) + 5g = 4(-4g + 5) - 23$

13 $-(5w + 14) + 9w = -3(5w + 1) - 11$

14 $-z - 2(5 - 4z) = -18 + 3(z - 1)$

15 $13p + 4(2p - 9) = -8 + 4(5p + 7)$

16 $8v + 7(3 - 2v) + 9 = 5 - 4(v + 5) + 8v$

17 $19 + 7y + 4(y - 3) = 5y - 4 - 2(y - 5)$

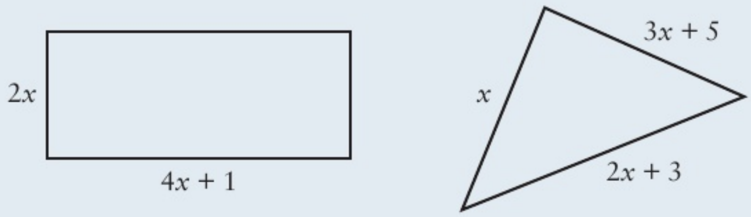
18 $27 - 2(4x - 3) + 6x = 14x - 3(3x + 1)$

For questions 19–28, work out a solution to each situation.

19 Twice a number reduced by seven is then multiplied by four. The result is −9. What is the number?

20 Twelve is added to five times a number and the result is multiplied by three. This is equal to eight less than four times the same number. What is the number?

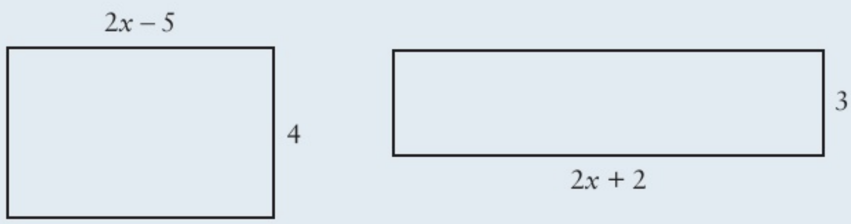
21 The rectangle and triangle shown below have the same perimeter. Find the perimeter.



22 The two rectangles shown below have the same area.

a Work out the dimensions of each rectangle.

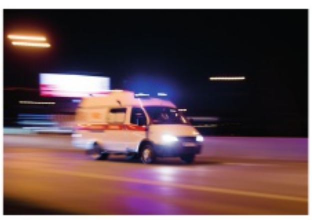
b Find the area of each rectangle.



23 Cally and her sister Bella are painting the walls in their house. Cally estimates that there is an area of $100\,m^2$ to paint. Cally paints at a rate of $0.4\,m^2$ per minute and Bella paints at a rate of $0.3\,m^2$ per minute. Bella starts painting 45 minutes after Cally started painting. How long will it take them to complete the work?

24 A computer shop sold 256 new computers and 212 refurbished computers this month. The number of new computers sold is decreasing by 14 computers each month, and the number of refurbished computers sold is increasing by 6 each month. If these trends continue, how many months will it be before the number of refurbished computers sold is twice the number of new computers sold?

25 An emergency vehicle leaves a fire station to travel to a fire. The vehicle travels at a speed of 60 km/h. A second vehicle leaves the same fire station 15 minutes later to go to the same fire. The second vehicle travels at a speed of 100 km/h. For how long, in minutes, does the first vehicle travel before the second vehicle catches it up?



Challenge Q25

Hint Q25

Make sure the units of time are the same. Use: distance = speed × time

26 A pet adoption fair is being held. 60 rabbits, cats and dogs are available for adoption. There are twelve more cats than rabbits, and there are twice as many dogs as there are cats and rabbits combined. How many of each type of animal are at the adoption fair?

27 A cruise ship and a fishing boat left Pier 42 at the same time, but travelling in opposite directions. The cruise ship travelled 12 km/h faster than the fishing boat. After 8 hours they were 592 km apart. How fast was each vessel travelling?

Challenge Q28

Reminder

Percentages should be expressed as decimals when performing calculations and writing equations.

28 Max has £20,000 to invest. He invests some of his money in an account that earns 6.5% simple interest and the remainder in an account that earns 8% simple interest. Assume that Max invests £x at 6.5% and the rest at 8%. After one year, he earns a total of £1498 in interest from both accounts. How much did he initially invest in each account?

6.4 Solving equations with fractions

There are many different ways of writing equations. We will now look at equations involving fractions. Equations involving fractions are helpful when modelling with rates or proportions, for example.

There are several possible approaches to solving equations that involve fractions. In this section, you will see some different methods, but you are encouraged to find other strategies that will also work.

Reminder

Recall that the top expression of a fraction is the called the **numerator**, and the bottom expression of a fraction is called the **denominator**.

Explore 6.4

A model of the Diamond Head Volcano in Hawaii has scale of 12 cm : 60 m. If the model is 46 cm high, can you determine the height of Diamond Head?

Worked example 6.9

Solve this equation for n.

$$\frac{5}{3}n + \frac{1}{6} = -4$$

Solution

In order to solve the equation for n, use inverse operations to isolate n on one side of the equation, and then solve for its value. Start by finding a common denominator, combine any like terms and then clear the fraction.

$\frac{5}{3}n + \frac{1}{6} = -4$	
$-\frac{1}{6} \qquad -\frac{1}{6}$	Subtract $\frac{1}{6}$ from both sides.
$\frac{5}{3}n = -4 - \frac{1}{6}$	Since 6 is the LCM of 3 and 6, we multiply by 6 to clear the fractions.
$6 \times \frac{5}{3}n = 6 \times \left(-4 - \frac{1}{6}\right)$	
$10n = -24 - 1$	Simplify.
$10n = -25$	
$\frac{10n}{10} = \frac{-25}{10}$	Divide both sides by 10.
$n = -\frac{5}{2}$	The solution is $n = -\frac{5}{2}$ or -2.5

To check the solution, substitute the value of n into the original equation and simplify.

$\frac{5}{3}\left(-\frac{5}{2}\right) + \frac{1}{6} = -4$	Substitute the solution into the equation, and follow the order of operations to simplify it.
	Combine the fractions and simplify the numerator on the left-hand side.
$\frac{-24}{6} = -4$	Simplify the fraction on the left-hand side by division.
$-4 = -4$	The left-hand side equals the right-hand side, so the solution is correct.

Alternative method

$$6 \times \left(\frac{5}{3}n + \frac{1}{6} = -4\right)$$
$$10n + 1 = -24$$

The lowest common multiple of the denominators (3 and 6) is 6, so we can clear all the fractions by multiplying **each term** of the equation by 6.

$$\underline{-1 \qquad -1}$$

Next, subtract 1 from both sides.

$$10n = -25$$

$$\frac{10n}{10} = \frac{-25}{10}$$

Then, divide both sides by 10 to isolate the variable.

$$n = -\frac{5}{2}$$

This gives the same solution: $n = -\frac{5}{2}$ or -2.5

Worked example 6.10

Ben and Carl are going to paint a house. Working alone, Ben can paint the house in 6 days, but Carl can paint the same house in 4 days. Working together, how long will it take Ben and Carl to paint the house?

Solution

This question is asking us to determine how many days it will take Ben and Carl to paint one house, if they work together. We know how long it will take each person individually. We can use that information to determine how long it will take when they work together.

Write expressions for each person's rate of working. Then combine the expressions to form an equation to represent the time needed to paint the house when they work together. Solve the equation and answer the question.

Let d = number of days needed to paint the house.

If it takes Ben 6 days to paint one house, then his rate of working is $\frac{1}{6}$ of a house per day. So, in d days, he can paint $\frac{1}{6}d$ houses.

Carl can paint the same house in 4 days, so his rate of working is $\frac{1}{4}$ of a house per day.

So, in d days, he can paint $\frac{1}{4}d$ houses.

Together, they need to paint 1 house in d days.

The equation to represent this is $\frac{1}{6}d + \frac{1}{4}d = 1$

$12 \times \left(\frac{1}{6}d + \frac{1}{4}d\right) = (1) \times 12$	The lowest common multiple of the denominators 6 and 4 is 12. Clear the fractions by multiplying each term on both sides of the equation by 12.
$2d + 3d = 12$	Then simplify each term.
$5d = 12$	Combine like terms on the left-hand side.
$\frac{5d}{5} = \frac{12}{5}$	Divide both sides by 5 to isolate the variable.
$d = \frac{12}{5} = 2.4$	The solution is $d = 2.4$

It will take 2.4 days for Ben and Carl to paint the house together.

Looking back, we can check the solution by substituting the value we found for d into the original equation and simplifying it.

$\frac{1}{6}(2.4) + \frac{1}{4}(2.4) = 1$ $0.4 + 0.6 = 1$	Substitute the solution into the equation, and follow the order of operations to simplify.
$1 = 1$	The left-hand side equals the right-hand side, so the solution is correct.

Reflect

There are often different ways to approach problem solving. You have seen two different methods in the examples above. Which method do you prefer? Can you think of other ways to solve these problems?

Practice questions 6.4

Solve these equations. Remember to check the solutions.

1 $\frac{h}{4} + 5 = -1$

2 $6 - \frac{2n}{5} = 2$

3 $-14 - \frac{3w}{4} = -11$

4 $\frac{4}{3} - \frac{10m}{3} = 8$

5 $\frac{3f}{16} - \frac{15}{16} = 3$

6 $\frac{p}{6} + \frac{5}{3} = 1$

7 $-\frac{57}{12} - \frac{3x}{4} = 5$

8 $\frac{q}{3} - \frac{7q}{5} = 16$

9 $\frac{15a}{16} - \frac{11a}{12} = -1$

10 $\frac{4}{3}(5 - 2g) = -4$

11 $-\frac{9}{2}(h - 6) = 18$

12 $\frac{7}{9}(3t - 6) = 14$

13 $\frac{5y + 14}{3} = \frac{3y - 4}{2}$

14 $\frac{12}{5} = \frac{5f - 5}{8 + 3f}$

15 $\frac{8}{13 - 5n} = \frac{9}{5 - 7n}$

16 $\frac{2}{3}p - 5 = 6 - \frac{1}{4}p$

17 $-14 + \frac{5}{4}b = \frac{11}{12}b - 11$

18 $\frac{7}{3}c - 21 = 19 + \frac{3}{7}c$

19 $\frac{2}{3}(1 - 4g) + 3 = \frac{4}{9}(g + 5) - 6g$

20 $\frac{7}{4}(2y + 1) - 5y = \frac{1}{6}(7 - 10y) + 2$

For questions 21–29, work out a solution to each situation.

21 $\frac{7}{8}$ of a number is 3 more than $\frac{5}{6}$ of the same number.
What is the number?

22 The denominator of a fraction is greater than the numerator by 5. If the numerator is increased by 12 and the denominator is decreased by 1, the number obtained is $\frac{5}{3}$. Find the original fraction.

23 Mary spent $\frac{1}{3}$ of her money on a book. She earned \$12 for mowing the lawn. She then had \$32. How much money did she start with?

24 Halfway through a baseball season, a pitcher has thrown 98 strikes out of 154 pitches.

a What is his current strike percentage?

b How many consecutive strikes does he need to throw in order to have a strike percentage of 75%?

Hint Q24b

Let x be the number of additional strikes the pitcher needs to throw. Then your denominator is the total number of pitches thrown and the numerator is the total number of strikes thrown.

25 Emily, Tara and Rita are helping to build sets for a theatre production at school. Working alone, Emily can complete the set in 15 hours, Tara can complete the set in 10 hours, and Rita can complete the work in 12 hours. Working together, how long will it take them to build the set?

26 A cyclist travels 25 km in the same amount of time it takes a second cyclist to travel 21 km. The first cyclist travels 4.5 km/h faster than the second cyclist.

Challenge Q26

a Use the formula distance = speed × time to write an equation for each cyclist.

b Using your equations from part a, determine the time it took each cyclist to travel these distances. Give your answer to the nearest minute.

27 Translate this statement into an equation and solve it:

One-third of five less than eight times a number is equal to seven more than twice the same number. What is the number?

28 The average cost in euros, C, to produce l units of a product is

Challenge Q28

$$C = \frac{400}{l + 15}$$

If the average cost per unit is €2, how many units were produced?

29 For 2005–2017, the average price p, in dollars, for a product can be modelled by $p = \frac{1024 + 9t}{120 - 4t}$, where $t = 0$ represents the year 2005. In which year was the average price \$10.70?

6.5 Graphing inequalities

An inequality is a number statement where the equals sign has been replaced by an inequality symbol. There are four different ways in which inequalities can be represented algebraically:

$<$	$>$	$\leq$	$\geq$
'less than'	'greater than'	'less than or equal to'	'greater than or equal to'

Inequalities can be shown graphically on number lines.

Graph 1

On Graph 1, the number 3 is *not* included as a solution. This is shown by an open circle. The arrow points to the right, so this represents numbers *greater than* 3.

Algebraically, this inequality is written as $x > 3$

Graph 2

On Graph 2, the number 3 *is* included as a solution. This is shown by a filled in circle. The arrow points to the right, so this represents numbers *greater than or equal to* 3.

Algebraically, it is written as $x \geq 3$

Explore 6.5

What is the difference between the solution of an equation and an inequality?

Write down a value that makes each equation and inequality in this table true. Use your findings to summarise the differences between solutions to equations and inequalities.

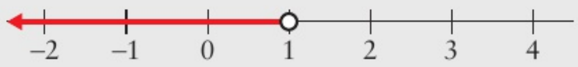
Equation	Inequality
$x - 3 = 10$	$x < 13$
$3x = -12$	$x \leq -3$
$x + 6 = 16$	$x > 10$
$2x - 1 = -7$	$x \geq -4$

Worked example 6.11

Show the inequality $x < 1$ on a number line.

Solution

In words, the inequality $x < 1$ states, 'all values less than 1'. On a number line, this can be represented with an open circle at 1 and an arrow pointing to the left.



Worked example 6.12

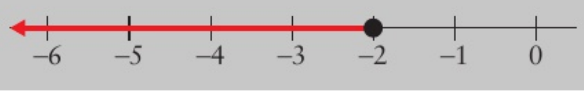
Graph the inequality $x \geq 0$ on a number line.

Solution

In words, the inequality $x \geq 0$ states, 'all values greater than or equal to 0'. On a number line, this can be represented with a closed circle at 0 and an arrow pointing to the right.

Worked example 6.13

Write an algebraic statement for the inequality shown by this number line.



Solution

There is a closed circle at -2, so -2 is included in the solution. The arrow is pointing to the left, so it shows values less than or equal to -2. The algebraic statement is $x \leq -2$

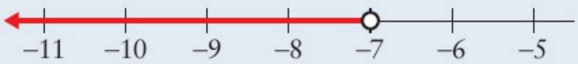
Look back at Explore 6.5. You should have discovered that equations in one variable have a fixed number of solutions (all the equations you have seen so far have one solution) and inequalities have solution sets that can contain an infinite number of solutions.

Practice questions 6.5

Show each of these inequalities on a number line.

1 $x > 2$	**2** $x \leq -5$	**3** $x \geq -1$	**4** $x > 3$
5 $x < 7$	**6** $x < -3$	**7** $x \leq -14$	**8** $x \geq -4$

Write an algebraic statement for the inequality shown by each of these number lines.

9


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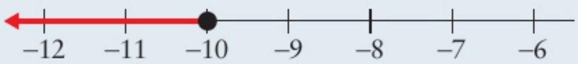

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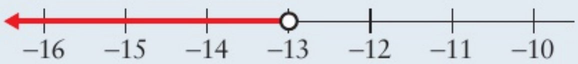


6.6 Solving inequalities

The process of solving inequalities is similar to the process of solving equations. We still use inverse operations to isolate the variable. And we can check our solution by substituting a value from the solution set into the original inequality. However, there are differences between the ways in which equations and inequalities are solved. The following investigation will help to uncover these differences.

Investigation 6.1

Each of the inequalities is this table is true. Carry out the given operation on both sides of the inequality, record your results, and state whether or not the inequality is still true.

1	$12 > 6$ Add 3	$-18 < -12$ Add 3	$18 > -12$ Add 3	$-6 < 12$ Add 3
2	$12 > 6$ Subtract 5	$-18 < -12$ Subtract 5	$18 > -12$ Subtract 5	$-6 < 12$ Subtract 5
3	$12 > 6$ Multiply by 3	$-18 < -12$ Multiply by 3	$18 > -12$ Multiply by 3	$-6 < 12$ Multiply by 3
4	$12 > 6$ Divide by 2	$-18 < -12$ Divide by 2	$18 > -12$ Divide by 2	$-6 < 12$ Divide by 2
5	$12 > 6$ Multiply by -2	$-18 < -12$ Multiply by -2	$18 > -12$ Multiply by -2	$-6 < 12$ Multiply by -2
6	$12 > 6$ Divide by -3	$-18 < -12$ Divide by -3	$18 > -12$ Divide by -3	$-6 < 12$ Divide by -3

If there are any inequalities that were no longer true after performing the given operation, what do they have in common? Generalise your results. What would you need to do to make these inequalities true after carrying out each operation?

Worked example 6.14

Solve this inequality for y. Then check the solution.

$19 - 3y > 7$

Solution

In order to solve the inequality for y, use inverse operations to isolate y on one side of the inequality, and then solve for its value. Start by combining any like terms.

$$\begin{array}{rcl} 19 - 3y & > & 7 \\ -19 & & -19 \\ \hline \end{array}$$

Simplify the inequality by subtracting 19 from both sides.

$$\frac{-3y}{-3} > \frac{-12}{-3}$$

Divide both sides by -3 to isolate the variable.

$$y < 4$$

The solution set is $y < 4$, or all numbers less than 4.

To check if the solution set satisfies the original inequality, choose a value that is part of the solution set and then substitute it into the original inequality.

Reminder

Because you are dividing by a negative number, you need to switch the sign of the inequality. Look back at your results from Investigation 6.1 to see why.

$19 - 3(2) > 7$	Substitute a value from the solution set into the inequality, and follow the order of operations to simplify. For example, let $y = 2$ be the test value as $2 < 4$
$13 > 7$	Subtract the terms on the left-hand side.
$13 > 7$	The inequality is still true so the test value chosen satisfies the original inequality. Hence, the solution set $y < 4$ is correct.

Worked example 6.15

Solve this inequality for n. Then check the solution.

$$\frac{12 + 11n}{5} \geq \frac{4n + 5}{2}$$

Solution

In order to solve the inequality for n, use inverse operations to isolate n on one side of the inequality, and then solve for its value. Start by clearing the fractions, using the distributive property, and then combine like terms.

$\frac{12 + 11n}{5} \geq \frac{4n + 5}{2}$	Start to simplify the inequality by multiplying both sides by both denominators to clear the fractions.
$2(12 + 11n) \geq 5(4n + 5)$	Expand the brackets on both sides.
$\begin{array}{r} 24 + 22n \geq 20n + 25 \\ -20n \qquad -20n \\ \hline \end{array}$	Subtract $20n$ from both sides.
$\begin{array}{r} 24 + 2n \geq 25 \\ -24 \qquad -24 \\ \hline \end{array}$	Then subtract 24 from both sides.
$\frac{2n}{2} \geq \frac{1}{2}$	Divide both sides by 2 to isolate the variable.
$n \geq \frac{1}{2}$	The solution set is $n \geq \frac{1}{2}$, or all numbers greater than or equal to $\frac{1}{2}$

To check whether the solution set satisfies the original inequality, choose a value that is part of the solution set and then substitute it into the original inequality.

$\frac{12 + 11(1)}{5} \geq \frac{4(1) + 5}{2}$	Substitute a value from the solution set into the inequality, and follow the order of operations to simplify it. For example, let $n = 1$ be the test value as $1 \geq \frac{1}{2}$
$\frac{23}{5} \geq \frac{9}{2}$	Simplify the numerators.
$\frac{23}{5} \geq \frac{9}{2}$, or $4.6 \geq 4.5$	Change the fractions to decimals to make it easier to compare them. The inequality is still true so the test value chosen satisfies the original inequality. Hence, the solution set $n \geq \frac{1}{2}$ is correct.

Worked example 6.16

A summer camp tracked the number of students who enrolled in their basketball and football programmes from 1995 to 2015. In 1995, the number of basketball students was 132. This number decreased by an average of 1.75 students each year. The number of football students enrolled in 1995 was 70. This number increased by an average of 2.5 students per year. For which years were there more students enrolled in basketball than football?

Solution

The question is asking us to compare two groups (basketball students and football students) over time and determine during which years one group (basketball students) had a higher number of participants than the other (football students).

To solve this problem, we will need to create an expression for the number of students enrolled in basketball and an expression for the number of students enrolled in football. Using these two expressions, we can create and solve an inequality to determine the years in which basketball had more students than football.

Let y represent the time in years and let $y = 0$ represent the year 1995.

An expression for the number of basketball students enrolled after y years is $132 - 1.75y$

An expression for the number of football students enrolled after y years is $70 + 2.5y$

To determine in which years there were more basketball students than football students, solve the inequality $132 - 1.75y > 70 + 2.5y$

$\begin{array}{r} 132 - 1.75y > 70 + 2.5y \\ \underline{+1.75y \qquad +1.75y} \end{array}$	Simplify the inequality by adding $1.75y$ to both sides.
$\begin{array}{r} 132 > 70 + 4.25y \\ \underline{-70 \qquad -70} \end{array}$	Subtract 70 from both sides.
$\dfrac{62}{4.25} > \dfrac{4.25y}{4.25}$	Divide both sides by 4.25 to isolate the variable.
$14.59 > y$	
$y < 14.59$	The solution set is $y < 14.59$, or all numbers less than 14.59

Looking back, how do we interpret the solution in the context of the problem? Does the answer make sense?

y represents the number of years since 1995. $y < 14.59$ means that during the first 14 years since 1995, that is, from 1995 to 2004, there were more students enrolled in basketball than in football.

We can also check the solution by substitution.

$132 - 1.75(\mathbf{14}) > 70 + 2.5(\mathbf{14})$	Substitute a value from the solution set into the inequality. For example, let $y = 14$ be the test value as $14 < 14.59$.
$132 - 24.5 > 70 + 35$	Follow the order of operations on each side in order to simplify the inequality.
$107.5 > 105$	The inequality is still true, so the test value chosen satisfies the original inequality. Hence, the solution set $y < 14.59$ is correct.

Reflect

By completing Investigation 6.1, you should have discovered that if you multiply or divide an inequality by a negative number you must also reverse the sign if the inequality is to remain true. Can you think of another way to explain why you reverse the inequality symbol when multiplying or dividing by a negative number?

Practice questions 6.6

Solve these inequalities. Remember to check the solutions.

1 $22 - 7y \leq 1$

2 $3a - 7 < 8$

3 $6h + 7 > 8h - 1$

4 $4c - 11 \geq 9c + 4$

5 $12 + 5b + 4 < 9b - 20 - 6b$

6 $34 + 12n > 16n + 14$

7 $6q - 23 \leq 15 - 2q$

8 $7w + 19 - 20w \geq 9 - 5w - 14$

9 $3(5 - f) > 4(f + 2)$

10 $-6(7 - 2l) < -(15 - 3l)$

11 $2(9z - 7) \leq 7(3z + 4)$

12 $18 + 4(7 + 5y) \geq 3(8y - 9) - 11$

13 $8(7 + 4r) - 11r < -3(6 - 5r) - 10$

14 $9x + 5(13 - 3x) \geq 2(5x - 11) - 1$

15 $\dfrac{5}{2d - 17} \leq \dfrac{2}{d + 3}$

16 $\dfrac{14 + 3x}{8} > \dfrac{2x + 5}{7}$

17 $\dfrac{4}{7} \geq \dfrac{2n - 3}{3n + 7}$

18 $\dfrac{8 - 5y}{6} + \dfrac{1 + 6y}{9} \leq 2$

Challenge Q18

19 $\dfrac{7(4 - 3x)}{6} + \dfrac{5(x + 1)}{3} > -1$

20 $\dfrac{2x - 3}{8} + \dfrac{2 - x}{4} < \dfrac{x + 1}{6}$

For questions 21–28, work out a solution to each situation.

21 Cally spends \$124 on material to knit scarves to raise money for the homeless. She sells each scarf for \$5.50. How many scarves does she need to sell to make a profit?

22 Hugo is considering joining a gym. The gym is offering a special discount to attract new members. For the first four months, the regular monthly fee will be reduced by €60. Hugo will join if the total cost of the trial period is less than €125. Write and solve an inequality to determine whether Hugo will join the gym.

23 Translate this statement into an inequality and solve it:

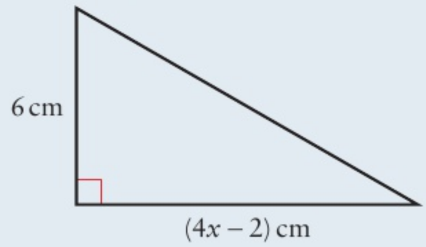
Seventeen more than four times a number is less than or equal to fifteen less than twice the same number. What is the number?

24 Translate this statement into an inequality and solve it:
Eighteen minus five times a number is greater than six more than the same number. What is the number?

25 Kylie is running a 20-km race. She runs the first 5 km in 27.6 minutes. She wants to finish the race in less than 2 hours and 15 minutes. Find the average speed (in km/min) at which Kylie needs to run the remaining distance.

26 Baljit needs to hire an electrician. BE Electrical charges a service fee of €75 plus an additional €28.50 per hour worked. 4U Electrical charges a service fee of €60 plus an additional €31.25 per hour worked. For how many hours of work is BE Electrical more expensive than 4U Electrical?

27 For the right-angled triangle shown find the values of x that make the area at least 66 cm^2.



28 Consider a rectangle that has a length three less than 4 times the width. Find the possible widths if the perimeter of the rectangle must be less than 224 cm.

Self assessment

- I can solve equations in one variable using addition, subtraction, multiplication and division.
- I can solve equations in one variable that involve brackets.
- I can solve equations in one variable that involve fractions.
- I can graph and solve inequalities in one variable using addition, subtraction, multiplication and division.
- I can apply problem-solving strategies and check my solution.
- I can set up equations and inequalities to model different real-world situations.
- I can use different techniques to solve equations and inequalities.
- I know that it is important to look back and check the solution to a problem.
- I know that it is possible to simplify and solve equations and inequalities in more than one way.

Check your knowledge questions

Solve these equations and inequalities. Remember to check the solutions.

1 $11 + 6d = -7$

2 $23 = 4f - 21$

3 $-37 = 15 - 13v$

4 $-8b - 19 = 17$

5 $9m - 5 = 6m + 19$

6 $16 + 7a = 12a - 4$

7 $13h + 20 = -22 + 6h$

8 $8 + 7x = -46 + 4x$

9 $2(3 + 5k) = 6(k + 3)$

10 $4(7 - 4u) = 3(8 - 5u)$

11 $2t + 5(t - 1) = 4 + 2(2t - 3)$

12 $12 - 3(6x - 5) = 7(3 - 2x) + 6$

13 $\frac{5r}{4} + \frac{1}{2} = -2$

14 $\frac{3y - 2}{6} = \frac{5}{3}$

15 $\frac{7p + 5}{5p - 9} = \frac{4}{3}$

16 $\frac{3n}{2} + \frac{2 - 9n}{8} = \frac{4n - 9}{6}$

17 $3x + 14 < 5x - 2$

18 $4(9 - 5x) \geq 7(3 - 5x)$

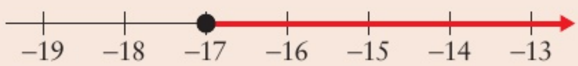
19 $\frac{4}{6w - 15} > \frac{3}{2w + 5}$

20 $\frac{30 + 8j}{3j - 8} \leq \frac{9}{2}$

21 Show the inequality $x > 19$ on a number line.

22 Show the inequality $x \leq -12$ on a number line.

23 Write an algebraic statement to represent the inequality shown on this number line.



24 Write an algebraic statement to represent the inequality shown on this number line.

For questions 25–32, work out a solution to each situation.

25 The sum of four consecutive even integers is 364. What are the numbers?

26 A recipe for pickles requires 2 kg of cucumbers, 1 kg of sugar and 500 ml of vinegar. You have 3 kg of cucumbers. How much sugar and vinegar should you use?

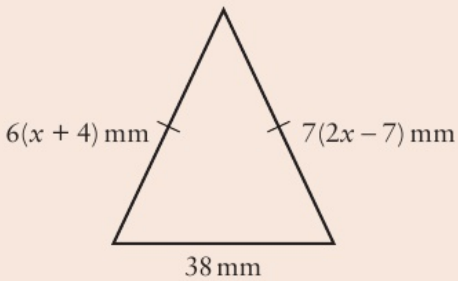
27 Companies that manufacture goods have fixed costs and variable costs. Fixed costs include salaries, utilities and maintenance on equipment.

Quali-Tea Manufacturing Company produces boxes of tea, and has fixed annual costs of \$350,000. The variable cost to produce one box of tea is \$3.25. They sell each box of tea for \$5.75. How many boxes of tea do they need to sell in order to break even?

28 Ghita wants to bring as many friends as possible to a comedy show. She has a total of €250 to spend. Transport to the show costs a total of €35 and each ticket costs €12. What is the greatest number of friends she can bring?

29 Maryam and Tarek drive their cars to the same concert. They leave home at the same time and travel the same distance. Maryam arrived in 2.5 hours and drove at an average speed of 100 km/h. Tarek arrived 20 minutes later. Find Tarek's average speed in km/h.

30 The triangle shown below is isosceles.
Work out the perimeter of the triangle.



31 In a chemical lab, one bottle contains 50% glycerine and another bottle contains 40% glycerine.

a 3 litres of the first glycerine solution and 6 litres of the second glycerine solution are poured together. Calculate the percentage of glycerine in the combined 9 litres.

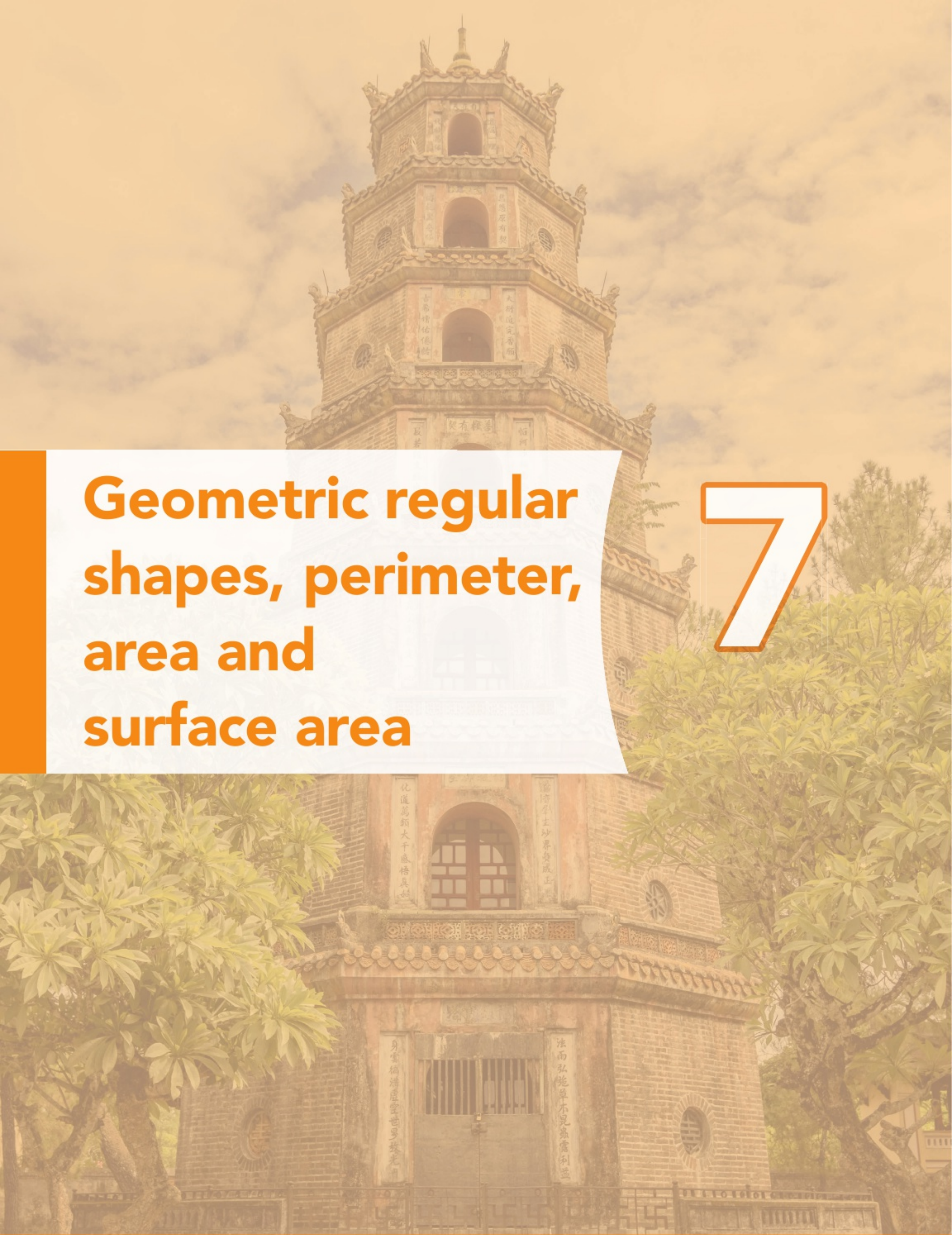
b How much of the 50% solution should be added to 3 litres of the 40% solution to get a solution that is 45% glycerine?

32 A company is studying the cost to remove a pollutant from the water near its site. The estimated cost, C (in thousands of dollars), to remove x percent (expressed as a decimal) of the pollutant can be modelled by the equation: $C = \dfrac{42x}{1.2 - x}$

a How much does it cost to remove 25% of the pollutant?

b How much does it cost to remove 50% of the pollutant?

c How much pollutant can the company remove if they spend \$84,000?



7 Geometric regular shapes, perimeter, area and surface area

7 Geometric regular shapes, perimeter, area and surface area

KEY CONCEPT

Form

RELATED CONCEPTS

Generalisation, Quantity, Space, Representation

GLOBAL CONTEXT

Orientation in space and time

Statement of inquiry

The form of objects in space can be represented and measured to understand landscapes around us.

Factual

- What is the perimeter of an object?
- What is the area of an object?

Conceptual

- How do you measure the area and perimeter of a polygon?
- How do you work out the surface area of a composite shape?

Debatable

- Can we precisely measure the surface area and volume of a natural landscape?

Do you recall?

Which of the statements in questions 1 to 4 are true and which are false?
Justify your answers.

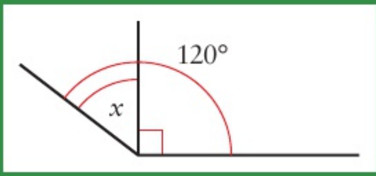
1 Angles of 2° and 178° are supplementary angles.

2 Angles of 43° and 57° are complementary angles.

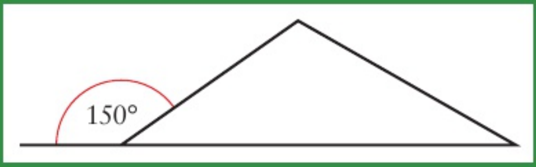
3 The sum of the interior angles of a triangle is 180°.

4 The sum of the interior angles of a rectangle is 360°.

5 Find the value of x in the diagram.

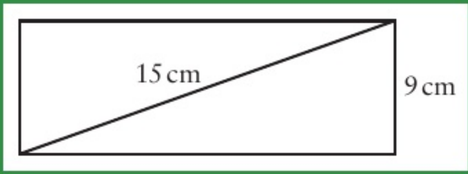


6 What are the measures of all the interior angles of the isoceles triangle below?



7 Sketch the graph of $2x - 3y = 6$ on a coordinate plane.

8 Find the area of the rectangle below.



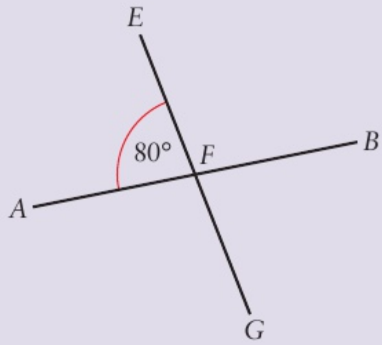
7.1 Angles and lines

In this section, you will refresh your knowledge of angles and lines.

Explore 7.1

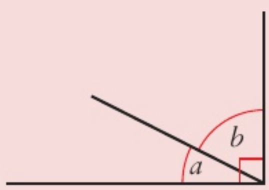
Consider the diagram below.

What do you observe? List all you know about this diagram.

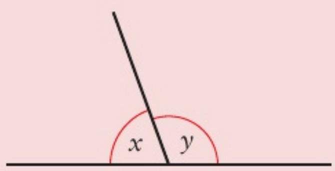


Fact

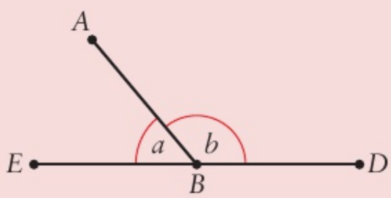
- If two adjacent angles add up to 90°, they are called **complementary angles**. In this diagram, a and b are complementary angles.



- Similarly, if two adjacent angles add up to 180°, then they are called **supplementary angles**. In this diagram, x and y are supplementary angles.



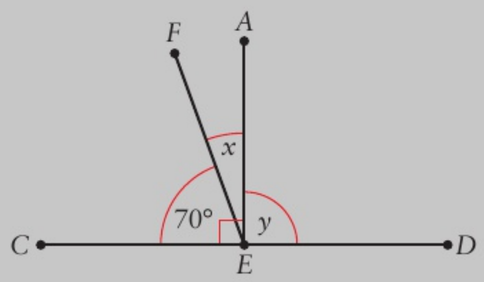
- If two adjacent angles are supplementary, they form a **straight angle**. $\angle EBD$ is a straight angle.



$a + b = 180°$

Worked example 7.1

Find the missing angles x and y in this diagram.



Solution

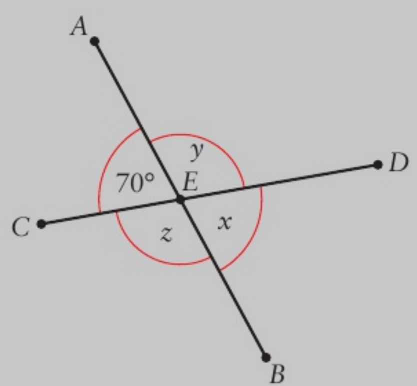
We need to use complementary and supplementary angle definitions. Note that $\measuredangle FEA = x$ and $\measuredangle FEC = 70°$ are adjacent complementary angles, and $\measuredangle AEC = 90°$ and $\measuredangle AED = y$ are adjacent supplementary angles.

$x + 70° = 90°$, $x = 90° - 70°$, so $x = 20°$
$90° + y = 180°$, $y = 180° - 90°$, so $y = 90°$

The calculated x and y values make sense. The line CD is a straight line and measures 180°, thus if we substitute values of x and y,
$\measuredangle FEC + \measuredangle FEA + \measuredangle AED = 180°$,
$70° + 20° + 90° = 180°$, which is true.

Worked example 7.2

Find the missing angles x, y and z.

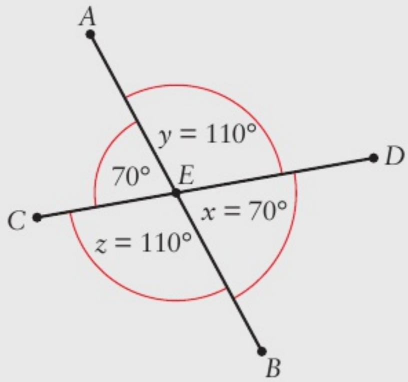


Solution

We can use the definition of vertically opposite angles. The pair consisting of $\measuredangle AEC = 70°$ and $\measuredangle DEB = x$, and the pair consisting of $\measuredangle CEB = z$ and $\measuredangle AED = y$ are vertically opposite angles and are therefore equal. Moreover, $\measuredangle AEC$ and $\measuredangle CEB$, and $\measuredangle AEC$ and $\measuredangle AED$ are supplementary angles.

Vertically opposite angles are equal: $x = 70°$ and $y = z$
Supplementary angles: $70° + z = 180°$, and $70° + y = 180°$
Therefore $y = z = 180° - 70° = 110°$

Looking back, if we substitute x, y and z values in the figure, the sum of each pair of adjacent angles is 180°, since they are supplementary angles.

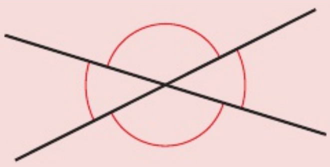


Reflect

Can we use coordinate geometry to work out this question?

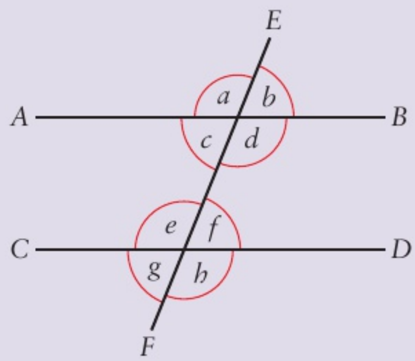
Fact

- If an angle is less than 90°, it is called an **acute angle**.
- If an angle is between 90° and 180°, it is called an **obtuse angle**.
- If an angle is more than 180° and less than 360°, it is called a **reflex angle**.



Explore 7.2

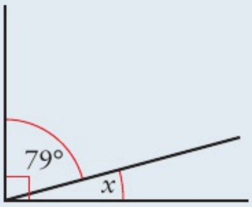
Parallel lines AB and CD are cut by a transversal. Aybuke claims that if one of the angles is known in this diagram, all of the other angles can be found. Is Aybuke's claim true?

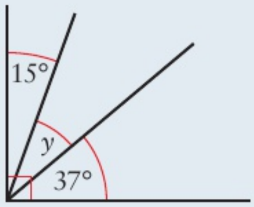


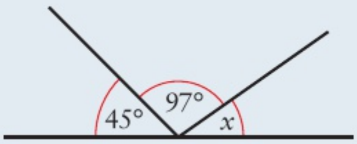
The girders of the Sydney Harbour Bridge demonstrate alternate angles and parallel lines.

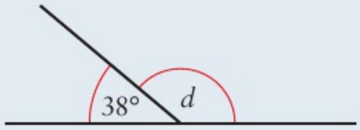
Practice questions 7.1

1 Find the values of the missing angles.

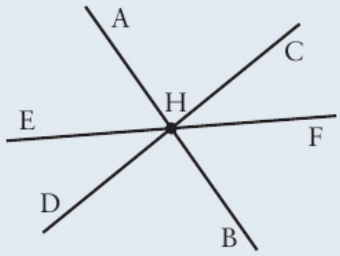
a


b


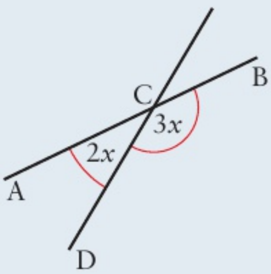
c


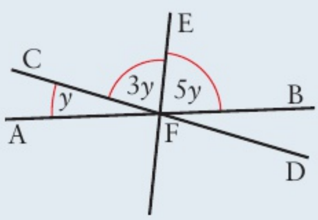
d


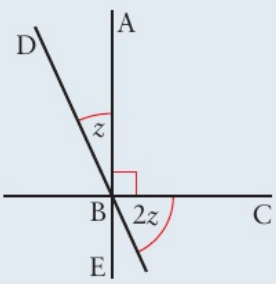
2 Write six possible pairs of adjacent angles in the diagram.



3 Find the missing angles. Show how you found your solution.

a


b


c


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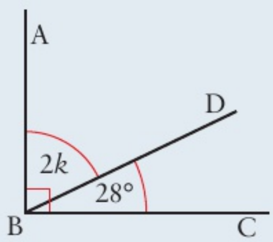
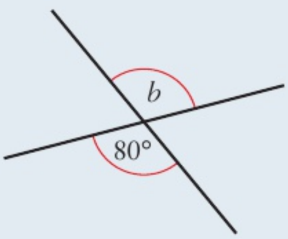
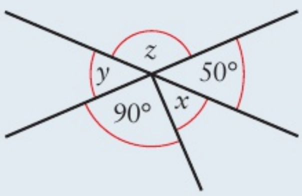
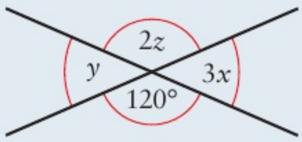
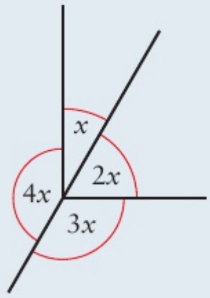


4 Calculate the values of the missing angles for each diagram and show how you reached your answer.

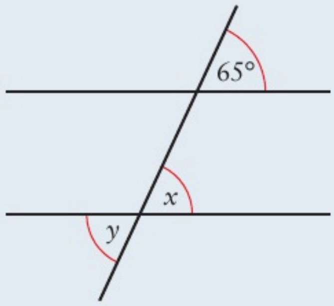
a


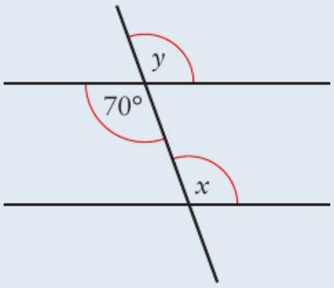
b


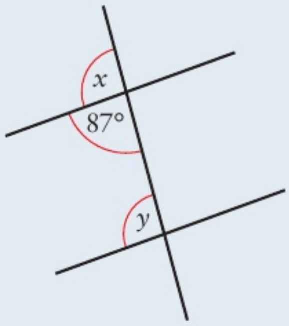
c


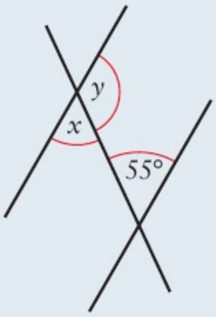
d


5 Calculate the missing angles, showing how you reached your answer, for each of the following pairs of parallel lines cut by transversals.

a


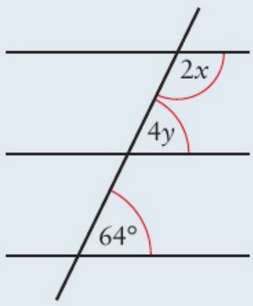
b


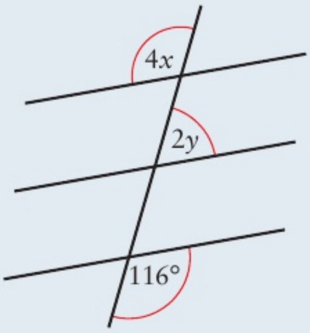
c


d


Challenge Q6

6 Calculate the missing angles for each of these parallel lines cut by transversals. Show how you reached each answer.

a


b


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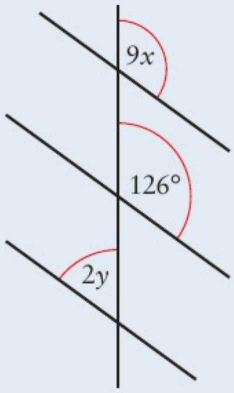

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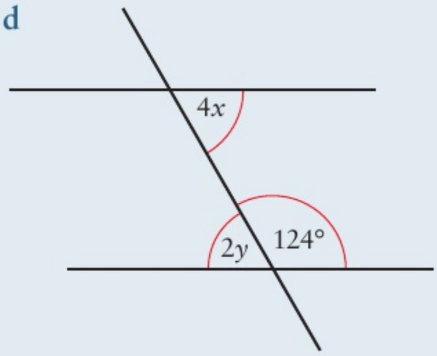
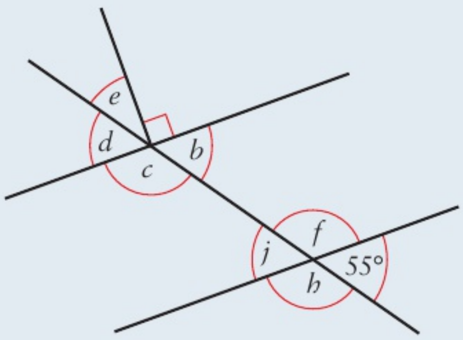


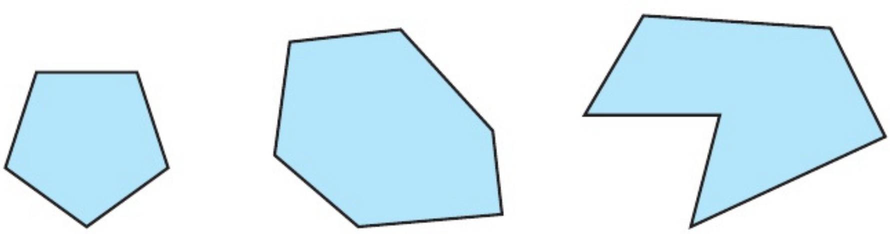
7 Calculate all the missing angles for the given parallel lines cut by a transversal. Show how you reached each answer.



7.2 Polygons

7.2.1 General polygons

When a plane figure is bounded by straight lines, it is called a **polygon**. If all of its sides and angles are equal, the polygon is called a **regular polygon**. Polygons are named according to the number of their sides: a triangle is a 3-sided polygon, a quadrilateral is a 4-sided polygon, a pentagon is a 5-sided polygon, a hexagon is a 6-sided polygon, and so on. If all of the interior angles of a polygon are acute or obtuse, it is called a **convex polygon**. If a polygon has a reflex angle in it, it is called **a concave polygon**.

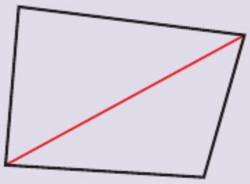


a regular pentagon a convex hexagon a concave hexagon

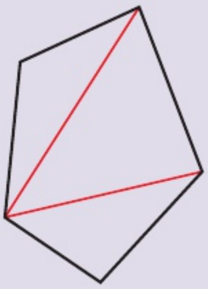
There are many polygonal shapes that can be observed in nature and human designs. For example, the pentagonal Mole Vanvitelliana in Ancona, Italy was built in the 18th century.

Explore 7.3

Consider the following polygons. Diagonals are joined from one of the vertices as shown.



Quadrilateral



Pentagon

Given that the sum of the interior angles of a triangle is 180°, can you work out the sum of the interior angles of each polygon?

If a polygon has n sides, how many triangles can be created using this method?

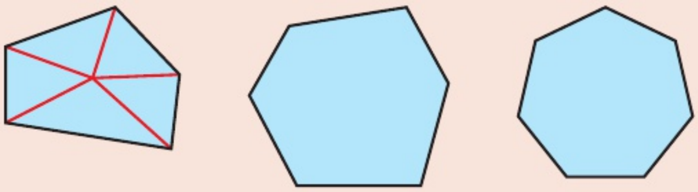
Can you generalise your findings for question 1?

Investigation 7.1

Exploring the sum of interior angles of a polygon

You can use available software or a GDC for this investigation.

1 From a point inside each of these polygons draw segments connecting that point to all of the vertices. The first one is done for you.



Pentagon Hexagon Heptagon

2 Sketch an 8-sided polygon, a 9-sided polygon, a 10-sided polygon and an 11-sided polygon. Divide each of these polygons into triangles as you did for question 1.

3 Use the results from questions 1 and 2 to complete this table.

Number of sides	5	6	7	8	9	10	11
Number of triangles							

4 What is the relationship between the two rows for the table in question 3?

5 As you see, for the pentagon we have 5 triangles. What is the sum of the angles of all the triangles?

6 What is the sum of the angles formed at the point inside the pentagon?

7 Can you now find the sum of the interior angles of the pentagon?

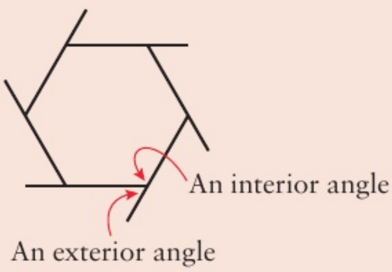
8 Can you answer questions 5–7 for the hexagon?

9 Repeat the same process for the other polygons if necessary. Can you generalise your conclusions to an n-sided polygon?

Investigation 7.2

The exterior angle sum of a convex polygon

1 At each vertex of the hexagon, a side has been produced to form an exterior angle. One interior angle/exterior angle pair is formed at each vertex to make a straight angle. How many straight angles does this hexagon have?



2 What is the sum of all the interior and exterior angles in this hexagon?

3 Use the answers to questions 1 and 2 to find the sum of all of the exterior angles.

4 Repeat steps 1 to 3 above for a regular pentagon.

5 What do the results above suggest about the sum of the exterior angles of any polygon?

6 Use the following to prove your observation in question 5:
Let the sum of the interior angles be I and the sum of the exterior angles be E.
For an n-sided polygon:
$I = 180(n - 2)$ and $I + E = 180n$ and thus, $180(n - 2) + E = 180n$
Solve the last equation for E.

Hint Q3

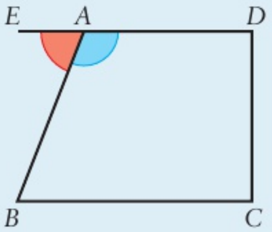
The sum of the interior angles of an n-sided polygon is $180(n - 2)°$.

Fact

- The sum of the interior angles of an n-sided polygon can be found using the formula $(n - 2) \times 180°$
- The sum of the exterior angles of **any** polygon is 360°

Hint

1 An exterior angle of a polygon can be formed by extending one of the interior angles at a corner. $\angle EAB$ is the exterior angle for the vertex A, $\angle BAD$ is the interior angle. Notice that $\angle EAB$ and $\angle BAD$ form a straight angle.



2 If there is a reflex angle in a polygon, then no exterior angle can be formed at that angle.

Worked example 7.3

For a regular 10-sided polygon:

a What is the sum of the interior angles?

b What is the sum of the exterior angles?

Solution

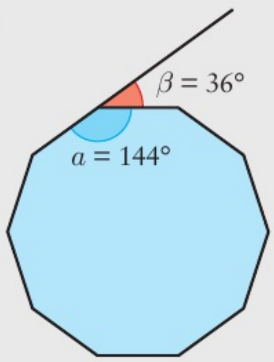
a We know that all of the angles and sides of a 10-sided polygon are equal.

Since we know that we have 10 triangles, and we know the sum of the angles at the centre, we can find the sum of the interior angles by subtracting the latter from the first.

The sum of the angles in 10 triangles is $180° \times 10 = 1800°$

The sum of the angles at the centre is 360°, and thus, the sum of the interior angles is $1800 - 360 = 1440°$

b Here is a diagram at one of the vertices.

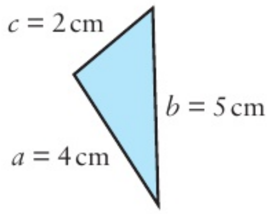
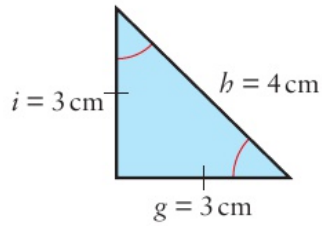
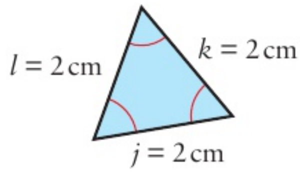


If the sum of all interior angles is 1440°, then each angle is 144°.

At each vertex, the exterior angle is supplementary to an interior angle, thus it has a measure of $180° - 144° = 36°$.
Therefore the sum of the exterior angles is $36 \times 10 = 360°$

Triangles

There are three types of triangle according to their sides:

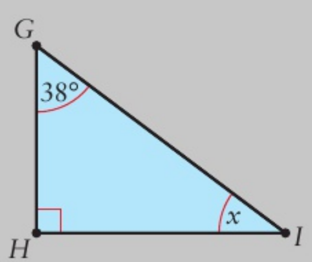
A scalene triangle has sides and angles that are all different.	An isosceles triangle has two equal length sides and angles.	An equilateral triangle has three equal length sides and all angles are 60°.
		

We can see triangles in the architecture of the house.

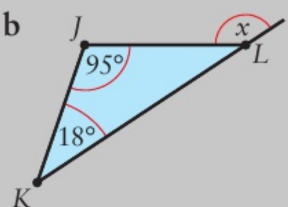
Worked example 7.4

What is the missing angle in each triangle? Classify each triangle and explain the reason.

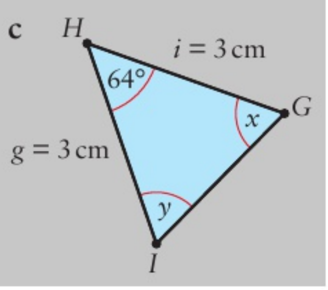
a



b



c



Solution

We know the sum of the interior angles of a triangle is 180° and we know how to classify triangles.

a $38 + 90 + x = 180$
$128 + x = 180$
$x = 180 - 128 = 52°$.
It is a right-angled triangle. Since all the angles are different, all the sides are also different and we can call the triangle scalene.

b The supplementary adjacent of x is $180 - x$

Therefore: $18 + 95 + 180 - x = 180$
$293 - x = 180$
$x = 113°$

The triangle is an obtuse-angled triangle and also scalene.

c Since two sides are equal, $g = i = 3$ cm, the triangle is an isosceles triangle. The base angles are

$x = y = (180 - 64) \div 2 = 58°$. It is also an acute-angled triangle.

If we substitute each angle found and sum the interior angles, they should add up to 180

a $38 + 90 + 52 = 180°$

b Supplementary angle of 113° is 67°, and $18 + 95 + 67 = 180°$

c $58 + 58 + 64 = 180°$

Since we know all the angles, the classification of each triangle is correct.

Quadrilaterals

Explore 7.4

Can you argue for or against the following statements?

All squares are rectangles, but not all rectangles are squares.

All parallelograms are rhombuses, but not all rhombuses are parallelograms.

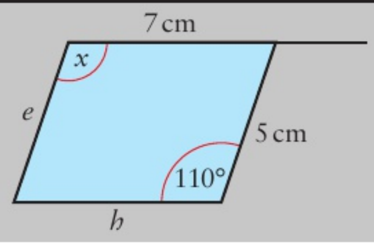
Fact

Here is a list of properties of the most used quadrilaterals which you should have seen before.

Properties	Square	Rectangle	Parallelogram	Kite	Rhombus	Trapezium
Opposite sides are parallel	Yes	Yes	Yes	No	Yes	No
Opposite sides are congruent	Yes	Yes	Yes	No	Yes	No
All sides are congruent	Yes	No	No	No	Yes	No
All angles are 90°	Yes	Yes	No	No	No	No
Opposite angles are congruent	Yes	Yes	Yes	No	Yes	No
Diagonals are congruent	Yes	Yes	No	No	No	No
Diagonals are perpendicular	Yes	No	No	Yes	Yes	No
Diagonals bisect each other	Yes	Yes	Yes	No	Yes	No
Only one pair of opposite sides are parallel	No	No	No	No	No	Yes

Worked example 7.5

Find the value of the missing angle and sides for the given parallelogram.



Solution

We are given the length of two sides and one of the interior angles. We need to find the two missing sides and the opposite angle.

We know opposite sides and angles in a parallelogram are equal.

$h = 7$ cm, $e = 5$ cm

$x = 110°$ since they are opposite angles.

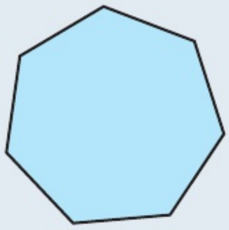
Looking back, opposite sides and angles of a parallelogram are congruent and our answer is accurate. That is, $h = 7$ cm, $e = 5$ cm, $x = 110°$

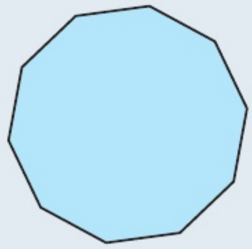
Practice questions 7.2

1 Find the sum of the interior angles of each n-sided polygon if n is:

a 5 b 9 c 14.

2 Find the value of each interior angle for these regular polygons.

a


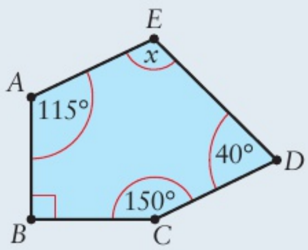
b


3 Find the sum of the interior angles of the regular polygon whose exterior angle is:

a 40° b 60° c 45°.

Challenge Q4

4 Find the value of the missing angles in each polygon. Give your reasons.

a


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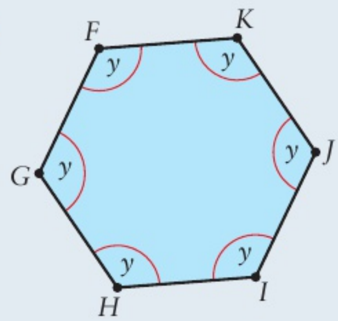

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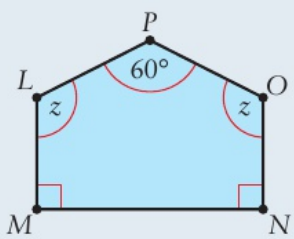
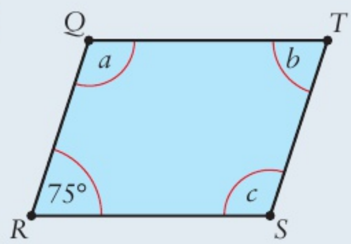
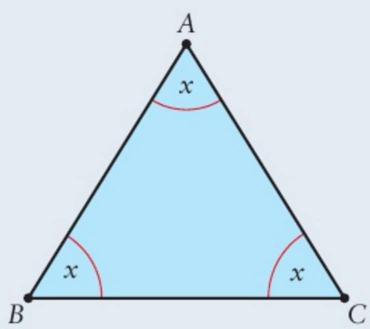


d


Hint Q4d

QRST is a parallelogram.

5 Find the value of the missing angles in each triangle. Give your reasons.

a


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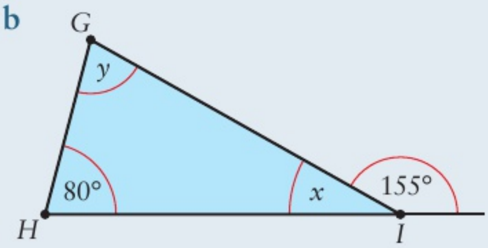
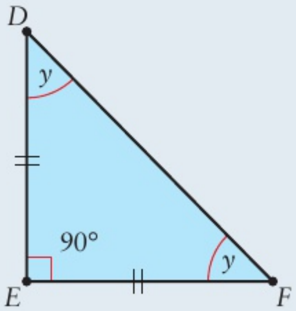


c 

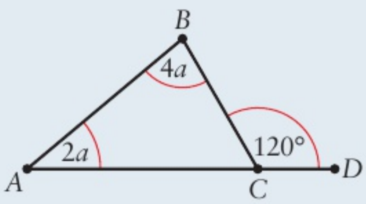
6 Identify whether each of the following statements is true.
Give your reasons.

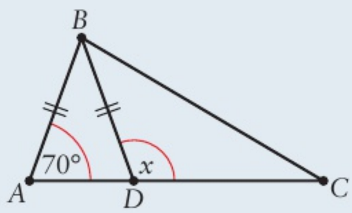
Challenge Q6

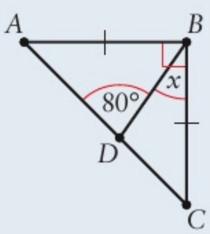
a A square is a rhombus.
b A rhombus is a square.
c A square is a rectangle.
d A rectangle is a square.
e A kite has two equal congruent pairs of sides.
f A trapezium can be isosceles.
g A kite is a rhombus.
h A rhombus is a kite.

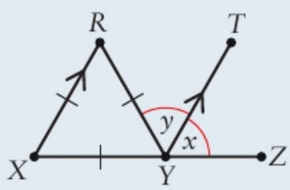
7 Show that the diagonals of a square bisect each other at right angles.

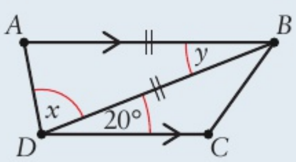
8 Find the value of each variable. Give your reasons.

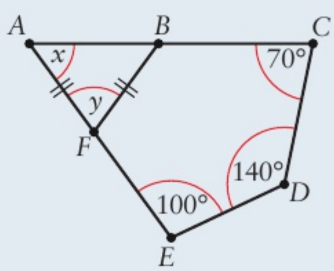
a 

b 

c 

d 

e 

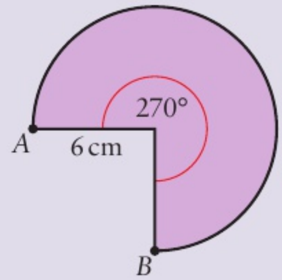
f 

7.3 Perimeter

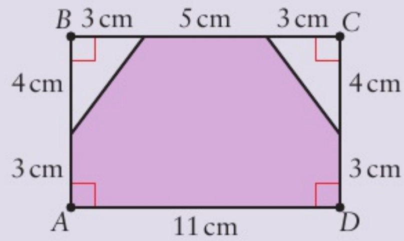
Explore 7.5

Can you find the perimeter of each of the shaded regions below?

a



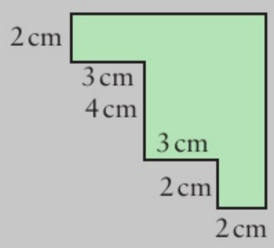
b



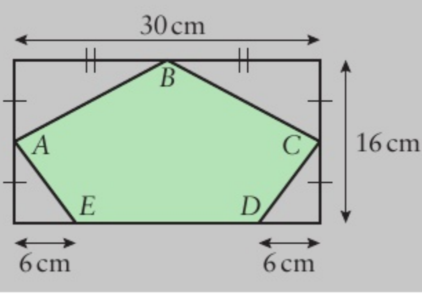
Worked example 7.6

Work out the perimeter of each shaded figure.

a



b



Solution

a This figure is a square with pieces cut out of it. We can see that the side measures 8 cm, by adding the segment lengths:

vertically: $2 + 4 + 2 = 8$

horizontally: $3 + 3 + 2 = 8$

The perimeter is the sum of all sides: $8 + 8 + 2 + 4 + 2 + 3 + 3 + 2 = 32$ cm

b This figure is a pentagon *ABCDE*.

ED can be found by subtracting the lengths of the two 6 cm corner segments from 30 cm:

$ED = 30 - 12 = 18$ cm

AB, *BC*, *AE* and *CD* are hypotenuses of their respective corner triangles.

Thus, using Pythagoras' theorem:

$AB = BC = \sqrt{8^2 + 15^2} = \sqrt{289} = 17\,\text{cm}$

and

$AE = CD = \sqrt{6^2 + 8^2} = 10\,\text{cm}$

Therefore, the perimeter of the pentagon is
$2 \times 17 + 2 \times 10 + 18 = 72\,\text{cm}$

Reflect

Can you justify the fact that the perimeter of the figure in part a above is the same as the perimeter of the whole square?

The perimeter of composite shapes

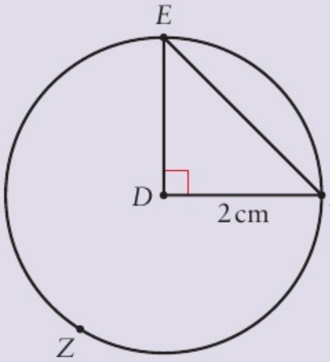
Explore 7.6

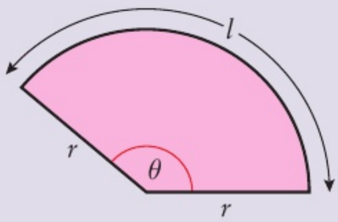
a Work out the circumference of the circle.

b Can you show why the length of the minor arc EF is $\frac{1}{4}$ of the circumference?

c What if the angle at the centre is not 90° but 120°?

d Can you generalise your response to question 3 for a part of a circle with radius r and a central angle θ?

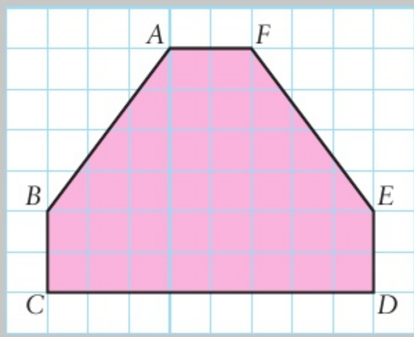




Composite figures are geometric shapes that are formed by bringing together more than one geometric shape. The perimeter of a composite geometric figure can be found by adding all sides of the shape.

Worked example 7.7

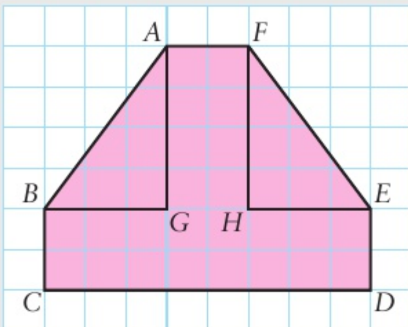
If *ABCDEF* is drawn to scale, find its perimeter.



Solution

We need to find the length of each side: *AB*, *BC*, *CD*, *DE*, *EF*, and *AF*.

All the sides except *AB* and *FE* are easily found by reading their coordinates on the grid. We can find *AB* and *FE* using Pythagoras' theorem by finding the right-angled triangles they belong to.



By drawing right-angled triangles *AGB* and *FHE* we see that they are congruent. The only length we need to calculate is *AB*, since the figure is drawn to scale.

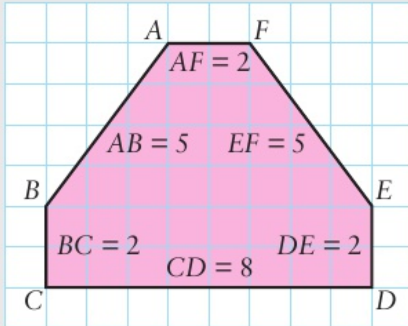
We can see that $AB^2 = 4^2 + 3^2$

Therefore, $AB = EF = 5$ units

So, $P = 5 + 2 + 8 + 2 + 5 + 2 = 24$ units

Looking back, the perimeter is the sum of the sides of *ABCDEF*.
The only values that we don't know are *AB* and *EF*. Those could also be found using GeoGebra. So, $AB = EF$.
If we add all the sides of the shape, we get the perimeter as 24 units.

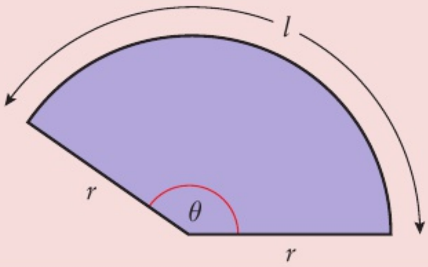
Perimeter of $ABCDEF = 24$



Fact

A **sector** is part of a circle and an **arc** is part of the circumference. So, the length of an arc l can be found by using the following formula:

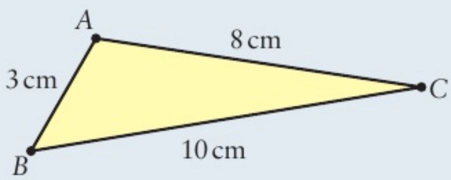
$$l = \frac{\theta}{360°} \times 2\pi r$$



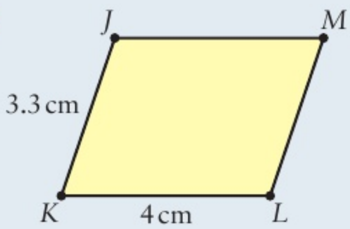
Practice questions 7.3

1 Find the perimeter of each of these shapes.

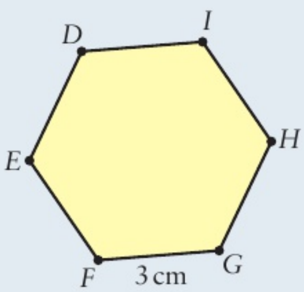
a



b

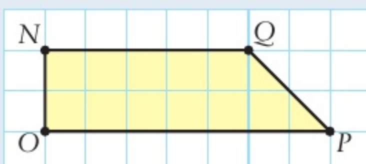


c



DEFGHI is a regular hexagon.

d



2 Find three possible rectangle dimensions that give a perimeter of 48 cm.

Hint Q1d

NOPQ is drawn to scale.

We see hexagons in nature.

Challenge Q2

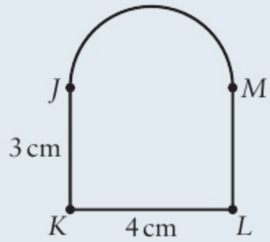
Challenge Q3

Reminder

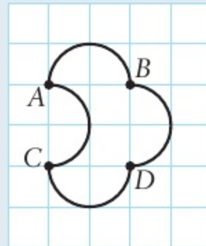
Remember to work out the missing sides to calculate the perimeter of each shape.

3 Find the perimeter of the shapes shown.

a



b



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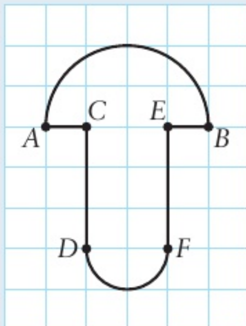

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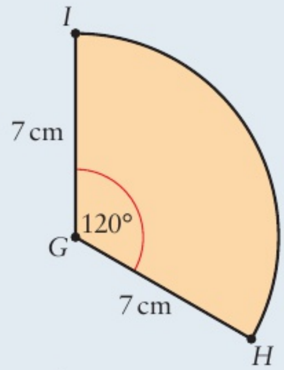
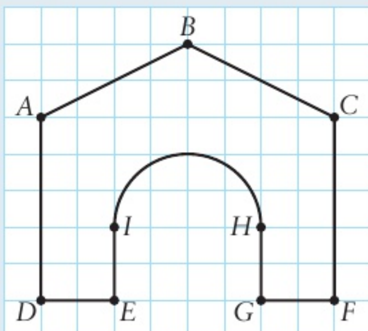


Challenge Q4

4 Find the perimeter of the composite shapes shown.

a



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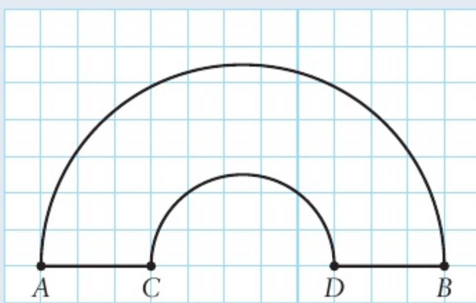

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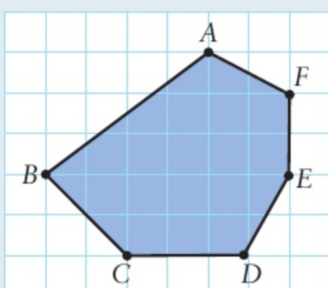
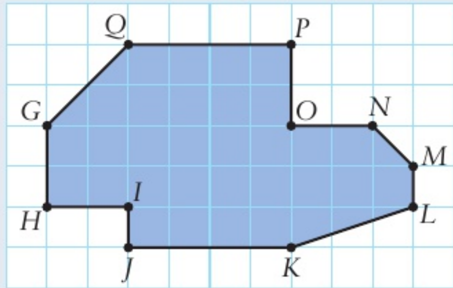
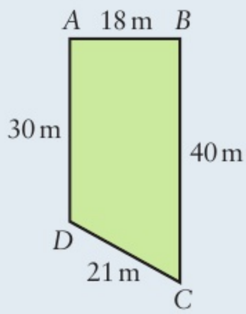
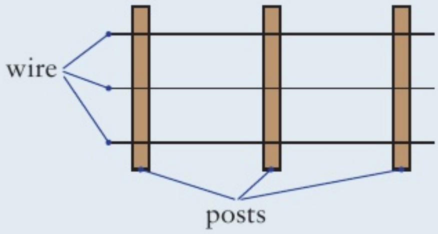


d 

5 A block of land (as shown in the diagram on the left) is to be fenced with a fence of the type shown on the right.



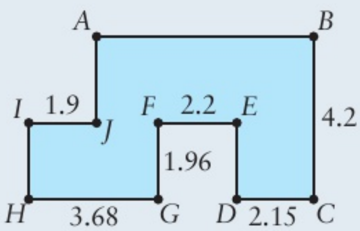


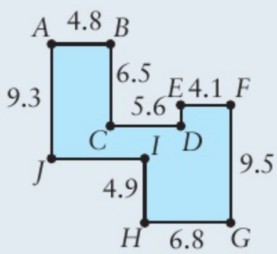
a Calculate the amount of wire needed to complete the fence.

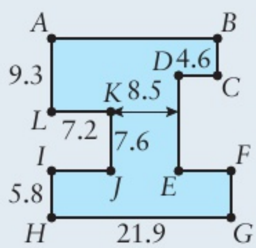
b If posts are to be placed at 3 m intervals (as a maximum), calculate the number of posts needed for the fence.

c Calculate the cost of the fence if the posts are €8.50 each and the wire is €0.95 a metre.

6 Find the perimeter of the following figures. All angles are right angles and all measurements are in metres.

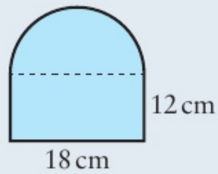
a 

b 

c 

7 Work out the perimeter of each of the following figures.

a



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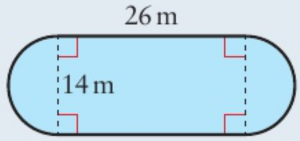

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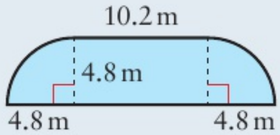

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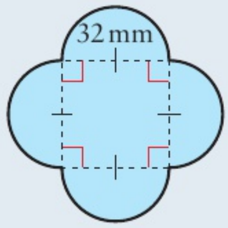

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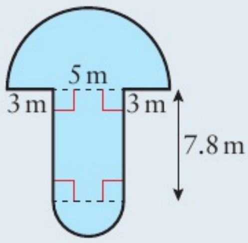

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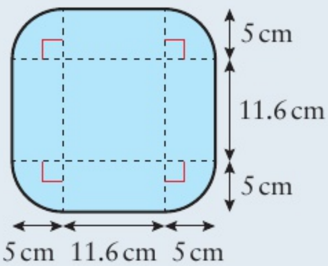
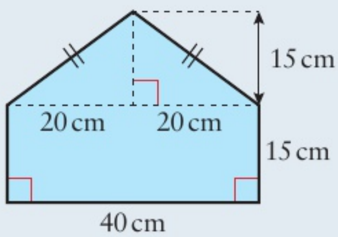


Challenge Q8

8 Work out the perimeter of each of these pentagons.

a



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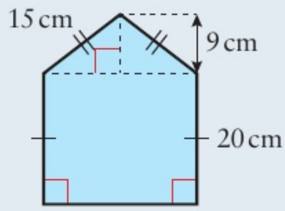

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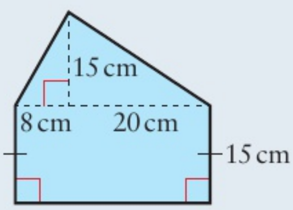
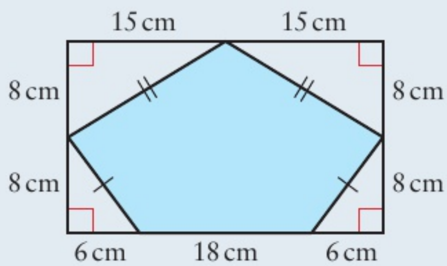


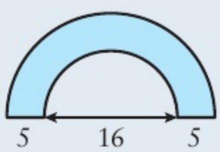
d



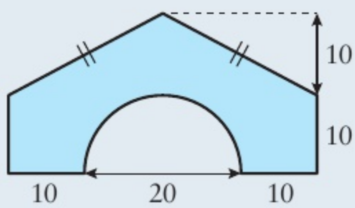
Challenge Q9

9 Work out the perimeter of the shapes below. Measurements are in centimetres. (Answer correct to one decimal place.)

a



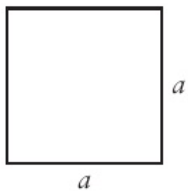
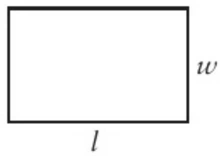
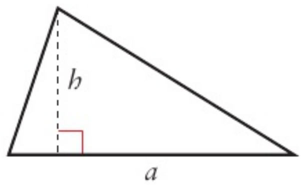
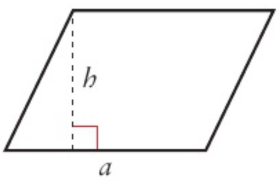
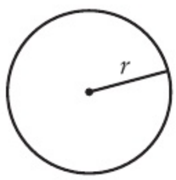
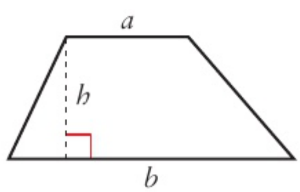
b



7.4 Area

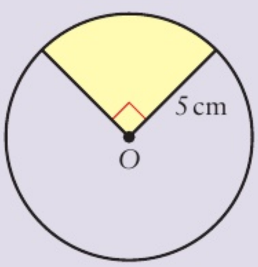
The area of a flat shape is the amount of space occupied by it. Area is measured by calculating how many squares the shape would cover.

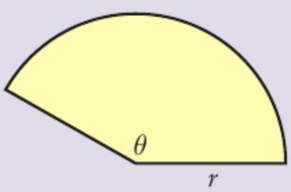
- 1 cm^2 is 1 cm × 1 cm
- 1 m^2 is 1 m × 1 m
- Area A is calculated by using the relevant area formula. Here are the formulae for some common shapes.

Shape		
 Formula $A = a \times a = a^2$ Square	 Formula $A = l \times w$ Rectangle	 Formula $A = \frac{1}{2}ah$ Triangle
 Formula $A = a \times h$ Parallelogram	 Formula $A = \pi \times r^2$ Circle	 Formula $A = \frac{1}{2}h(a + b)$ Trapezium

Explore 7.7

Can you show why the area of this quadrant is $\frac{1}{4}$ the area of the circle?





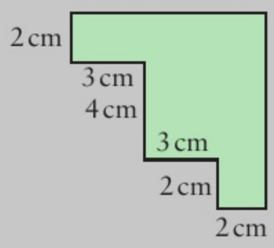
What if the angle at the centre is not 90° but 120°?

Can you generalise your response to questions 1 and 2 for a sector of a circle with radius r and a central angle θ?

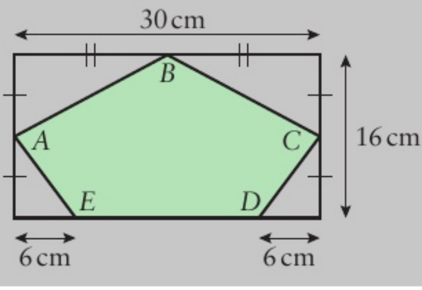
Worked example 7.8

Work out the area of each of the shaded figures from Worked example 7.6.

a

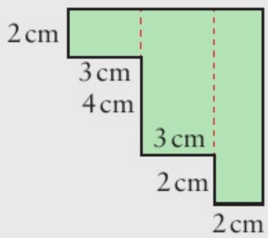


b



Solution

a We know that the figure is a square with pieces cut out of it.
By subdividing it into small rectangles, we can find the area by adding the areas of the small pieces:



We now have three rectangles of the following measurements:

3 cm by 2 cm, 3 cm by 6 cm, 2 cm by 8 cm

Thus, the total area $= 3 \times 2 + 3 \times 6 + 2 \times 8 = 40\,\text{cm}^2$

b The area of pentagon *ABCDE* can be found by subtracting the areas of the corner triangles from the area of the largest rectangle:

area of rectangle $= 30 \times 16 = 480\,\text{cm}^2$

corner triangles areas $= 2\left(\frac{1}{2} \times 6 \times 8 + \frac{1}{2} \times 15 \times 8\right) = 168\,\text{cm}^2$

area of pentagon $= 480 - 168 = 312\,\text{cm}^2$

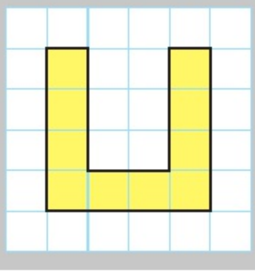
Reflect

Each part of Worked Example 7.8 can be done in at least two different ways. Can you work out one for each?

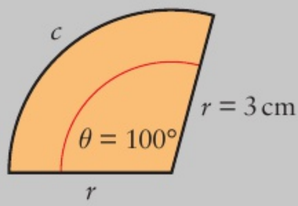
Worked example 7.9

Find the area of the following shapes. One shape is drawn on a grid.

a



b



Solution

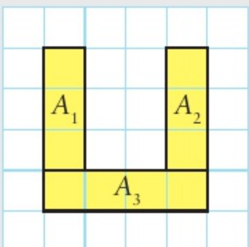
Make a plan

a The area of this shape can be found either by adding the sub-parts or by subtracting the missing part of the shape from a larger complete shape.

b The area of this shape can be calculated by using the formula for the area of a sector.

Carry out the plan

a Let's do it by adding the areas of the sub-parts.



$A = A_1 + A_2 + A_3 = 3 + 4 + 3 = 10$ units^2

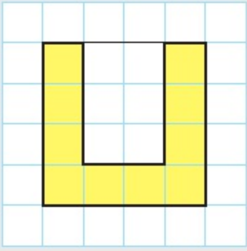
b $A = \dfrac{100°}{360°} \times \pi \times 3^2 = 2.5\pi\,\text{cm}^2$

Fact

The area of a composite shape can be calculated either by adding the areas of each sub-divided part, or by subtracting the missing part of the shape from a larger complete shape.

Look back

a We can check the area using the method of subtracting the missing part of the shape from a larger complete shape.



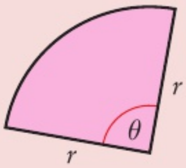
The shaded area is the difference between the complete square and the area of the unshaded rectangle. Thus,

$A = 16 - 6 = 10$ units^2

b The area of a sector is $\frac{100°}{360°} \times \pi \times 3^2 = 2.5\pi$ cm^2

Fact

The area of a sector can be found by using the following formula.

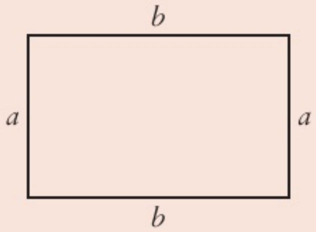


$A = \frac{\theta}{360°}\pi r^2$

Investigation 7.3

Do you think there is any connection between the perimeter and the area of a rectangle?

Consider a rectangle with a perimeter of 30 cm.



Answer the following questions:

1 Find four different pairs of (a, b) for a rectangle with perimeter 30 cm. Include one case where $a = b$

2 Show that the perimeter is 30 cm in each of your examples.

3 Calculate the area of each rectangle for different values of a and b.

4 Record your findings in the table.

Rectangle	a	b	Perimeter	Area
Rectangle 1				
Rectangle 2				
Rectangle 3				
Rectangle 4				

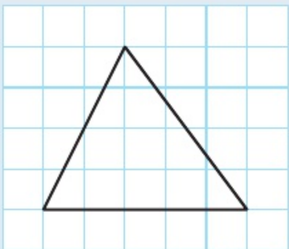
5 Can you predict which rectangle will have the greatest area?
Justify your prediction.

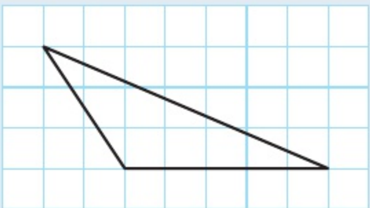
6 Find the greatest area for a rectangle that has a perimeter of 24 cm.

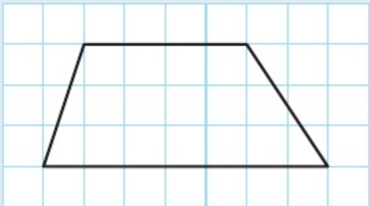
7 Generalise your observations from questions 5 and 6.

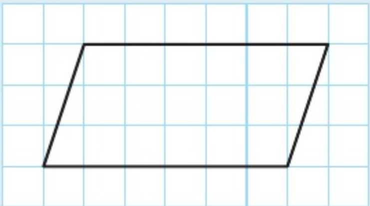
Practice questions 7.4

1 Calculate the areas of these shapes.

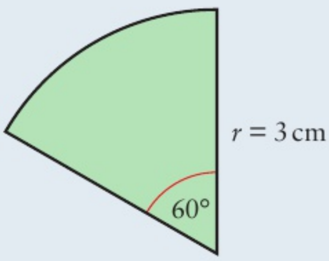
a 

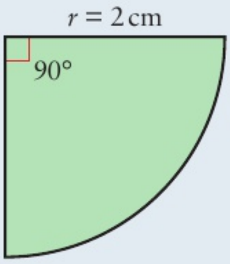
b 

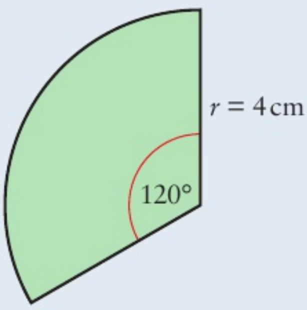
c 

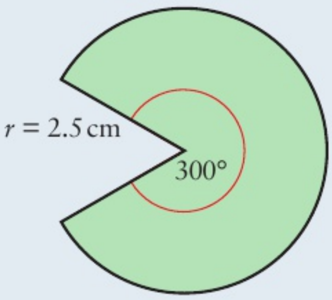
d 

2 Calculate the areas of these sectors.

a 

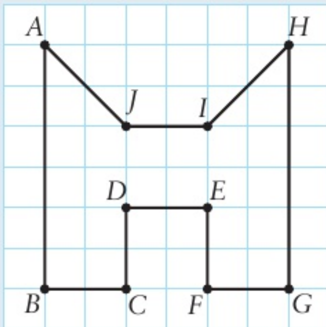
b 

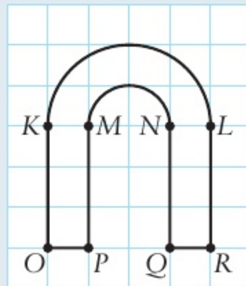
c 

d 

Challenge Q3

3 Find the area of each of these shapes that are drawn on grids.

a


b


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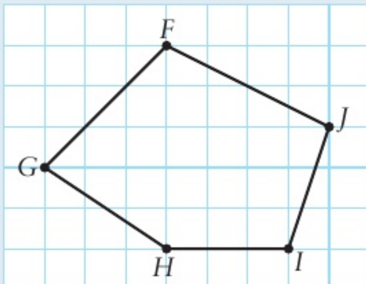
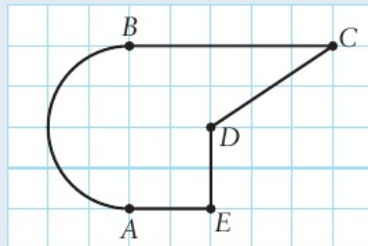
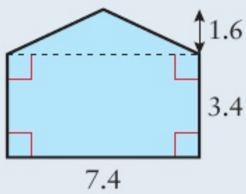
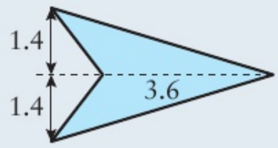
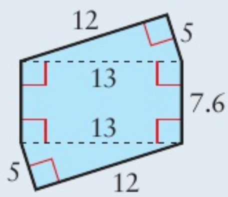


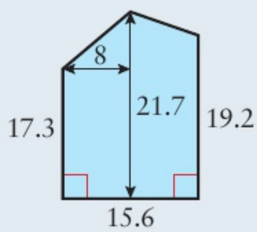
d


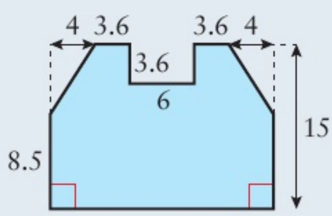
4 Work out the area of each figure. All measurements are in metres.

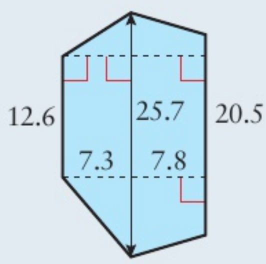
a


b


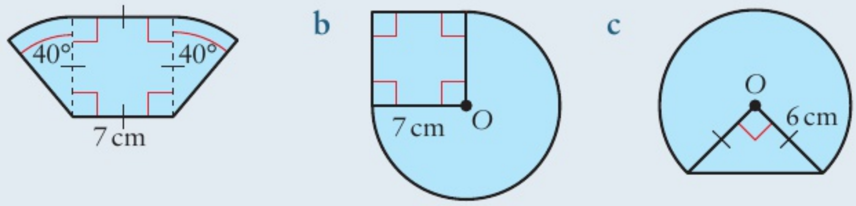
c


d


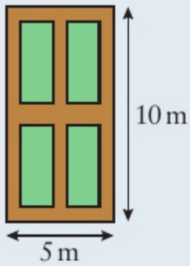
e


f


5 Find the area of these figures correct to the nearest cm^2.



6 The diagram shows four garden beds, each 3.4 m long and 1.8 m wide, surrounded by paving bricks. Find:



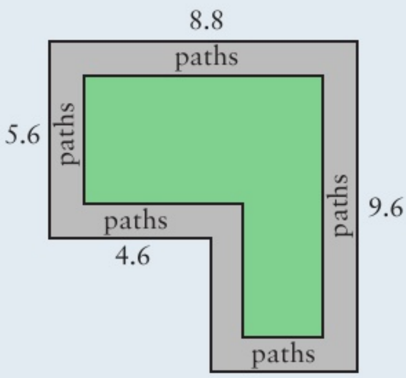
a the area of the garden beds

b the paved area

c the number of bricks needed to pave the area (rounding up to the next hundred) if a brick is 225 mm by 112 mm.

Challenge Q7

7 Find the cost of concreting the paths in the diagram if concrete costs \$50 per m^2. All paths are 0.9 m wide and all measurements are in metres.



7.5 Surface area

The surface area of an object can be calculated by adding the areas of its faces. As we live in a three-dimensional world, we commonly deal with solid objects that are in fact prisms or cylinders. The classrooms we work in are nearly all rectangular prisms, while water tanks and swimming pools are often cylindrical. Because we often come into contact with many of these objects, a knowledge of surface area can be extremely useful. Architects, painters, builders and many others require a sound knowledge of surface area to enable them to carry out their work.

Explore 7.8

A cylindrical water tank (with a top) has a radius of 2.1 m and a height of 3.2 m. The tank is to be painted (inside) with a rust preventive paint.

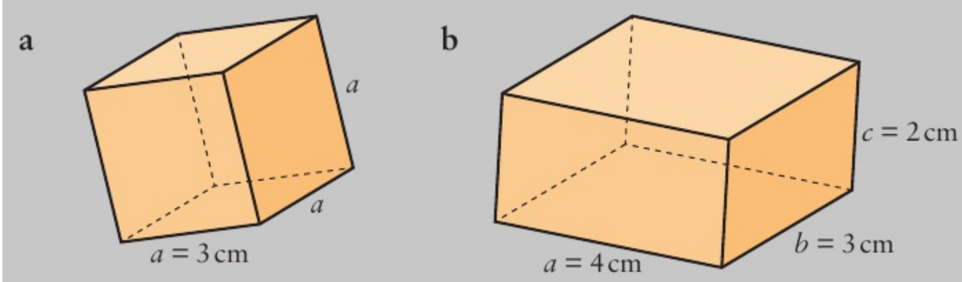
Can you find:

a the total area that is to be painted (correct to one decimal place)

b the total cost of the paint if it costs \$10.25 per m^2.

Worked example 7.10

Find the surface area of these two shapes.



Cube

Rectangular prism

Solution

We are asked to find the surface area of a cube and a rectangular prism, and we know the length of all sides.

A cube has six identical faces, and each one is a square. A rectangular prism has three pairs of congruent rectangular faces. We have the necessary information to find the area of each face.

a The area of each square face of the cube is 3 cm × 3 cm = 9 cm^2, so the surface area of the whole cube is 6 × 9 cm^2 = 54 cm^2

b We have three pairs of identical rectangular faces (A_1, A_2, A_3) that are opposite each other.

A_1 = 4 cm × 2 cm = 8 cm^2, A_2 = 4 cm × 3 cm = 12 cm^2
and A_3 = 3 cm × 2 cm = 6 cm^2

Therefore, the surface area = $2A_1 + 2A_2 + 2A_3$ = 2(8 + 12 + 6) = 52 cm^2

The cube has six faces, each being a square measuring 3 cm by 3 cm.

Surface area = 6(3 cm × 3 cm) = 54 cm^2

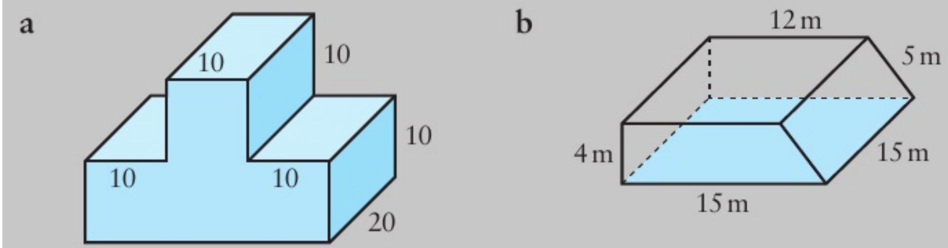
The rectangular prism has surface area = 2(8 cm + 12 cm + 6 cm)
= 2 × 26 cm = 52 cm^2

Reminder

Closed prisms have a top; open prisms have no top. For example, if the cube in the worked example was an open cube, the surface area would be 5 × 9 cm^2 = 45 cm^2

Worked example 7.11

Find the surface area, SA, of the solids shown.



Solution

a The composite shape is a combination of two rectangular prisms. The front and back consist of a 30 by 10 rectangle and a 10 by 10 rectangle. The base is a 30 by 20 rectangle. The sides and top consist of seven 20 by 10 rectangles.

$$SA = \underbrace{(20 \times 30)}_{\text{base}} + \underbrace{7 \times (20 \times 10)}_{\text{sides \& top}} + \underbrace{2 \times (10 \times 10 + 10 \times 30)}_{\text{front \& back}}$$

$$= 600 + 7 \times 200 + 2 \times 400 = 2800 \text{ units}^2$$

b The shape is a trapezoidal prism.

The front and back faces are right trapeziums with bases 15 and 12 and a height of 4. The sides are rectangles with different dimensions: the base is a 15 by 15 square, the top is a 15 by 12 rectangle, and the other faces are a 5 by 15 and a 4 by 15 rectangle.

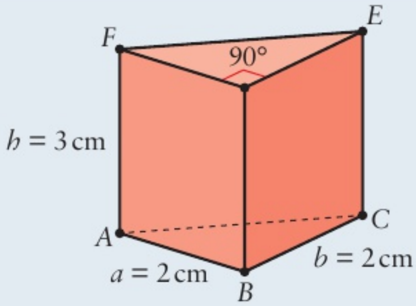
$$SA = \underbrace{2 \times \frac{1}{2}(15 + 12) \times 4}_{\text{front \& back}} + \underbrace{15 \times 15}_{\text{base}} + \underbrace{15 \times 12}_{\text{top}} + \underbrace{5 \times 15}_{\text{side}} + \underbrace{4 \times 15}_{\text{side}} = 648\,\text{m}^2$$

Reflect

Could you have approached these two problems differently?

Practice questions 7.5

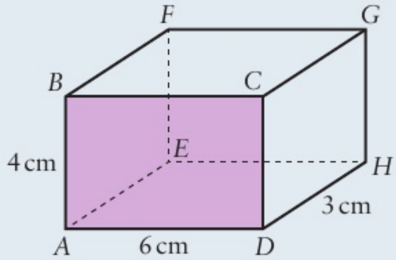
1 Find the surface area of the triangular prism.



Hint Q2

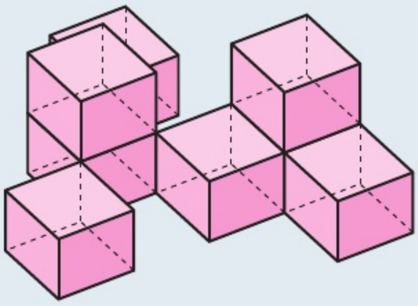
The open side is *ABCD*, which is the shaded side in the diagram.

2 Calculate the surface area of the open rectangular prism *ABCDHGFE*.



Challenge Q3

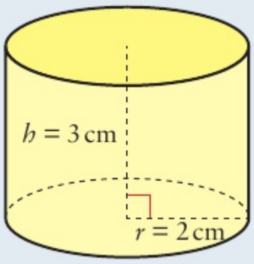
3 The solid shown has been built up by using 1 cm³ cubes.
Calculate the total surface area of the solid.



Hint Q4

The surface area of a cylinder is $2\pi rh + 2\pi r^2$, where r is the radius and h is the height of the cylinder.

4 Find the surface area of the cylinder.



Challenge Q5

Hint Q5

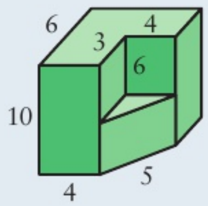
The swimming pool should be considered as an open rectangular prism.

5 An Olympic swimming pool is in the shape of a rectangular prism, and the dimensions of the pool are 50 m × 25 m × 2 m. The pool is to be tiled for the New Year games.

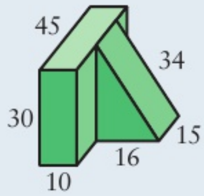
Calculate the total cost if the tiling is €35 per square metre.

6 Work out the surface area of the following solids. (All measurements are in centimetres.)

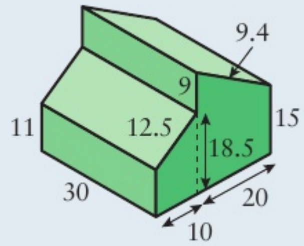
a



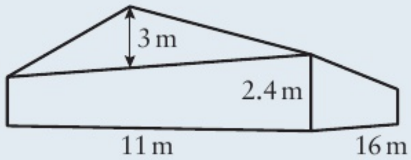
b



c

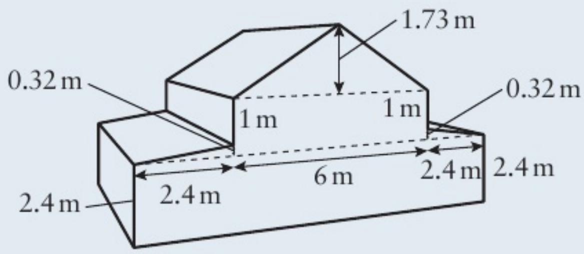


7 A marquee is in the shape of a pentagonal prism. Use the dimensions shown to calculate the surface area. (There is no floor.)



Challenge Q7

8 A barn is made from aluminium. Calculate the area of metal used in its construction. (Assume it has a total window area of $8.8\,\text{m}^2$ and a length of 7.2 m.)



Challenge Q8

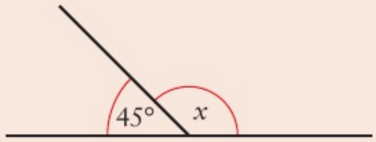
Self assessment

- I can work with supplementary and complementary angles.
- I can work with alternate, co-interior and corresponding angles.
- I can work with the sum of interior angles of a polygon.
- I can work with the sum of the exterior angles of a polygon.
- I can classify a polygon according to its sides.
- I can classify triangles according to their sides and angles.
- I can find the perimeter of a polygon.
- I can solve perimeter problems for composite shapes.
- I can find the areas of triangles, quadrilaterals and circles.
- I can calculate the area of composite shapes.
- I know that there is a relationship between the perimeter and area of a rectangle.
- I can find the surface area of prisms.
- I can find the surface area of composite solids.
- I can use a GDC and software programs to draw polygons.
- I can draw 3D solids with available software.

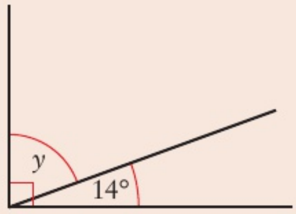
Check your knowledge questions

Calculate the size of the missing angles in questions 1–4 below.

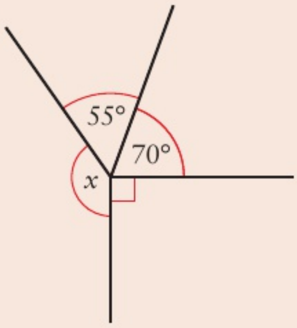
1 a



b



2 a



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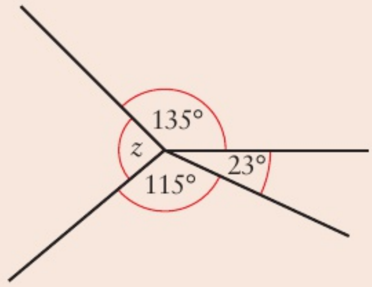
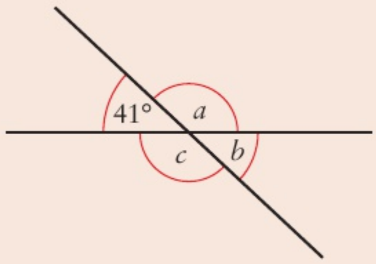


3 a



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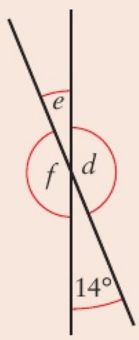
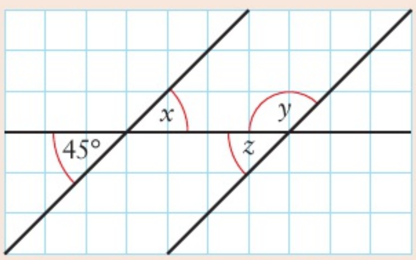


Hint Q4

For each pair of parallel lines, a transversal cuts them both.

4 a



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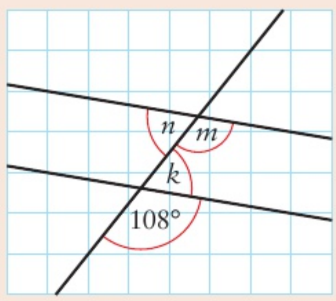


Challenge Q5a

5 a Calculate the sum of the interior angles of a pentagon.

b Write down the sum of the exterior angles of an 11-sided polygon.

c If the sum of the interior angles of a regular polygon is 1260°, calculate the size of one of its exterior angles.

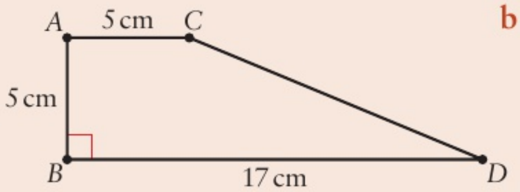
d Calculate the number of sides of a regular polygon if one of its exterior angles is 36°.

For questions 6–9, calculate the perimeter or circumference of each shape.

6 **a** The circumference of a circle where $r = 2.5$ cm

b The circumference of a circle where $d = 16$ m

7 **a**



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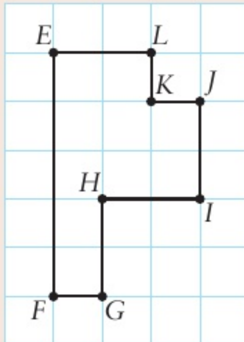
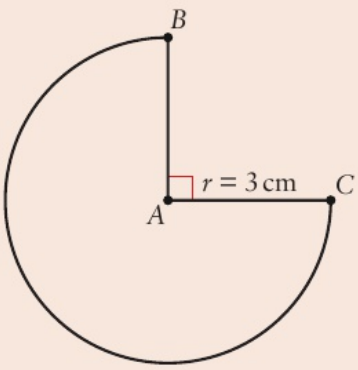


8 **a**



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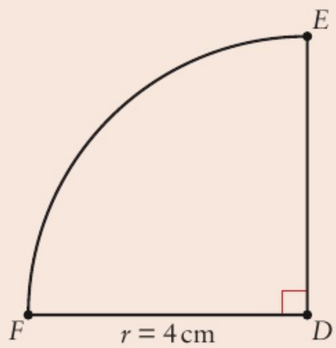
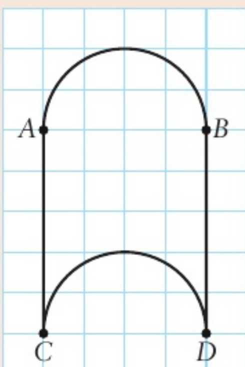
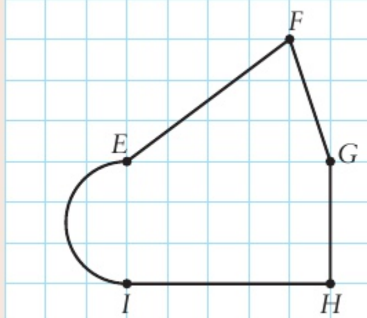


9 **a**

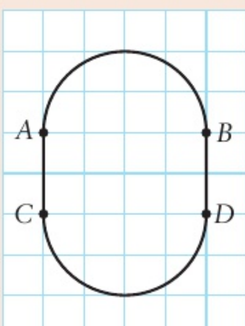


b

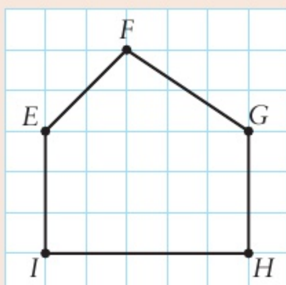


For questions 10–13, calculate the area of each shape.

10 **a**



b

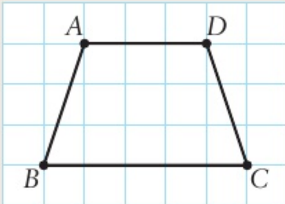


11 Calculate the area of the sectors in question 8.

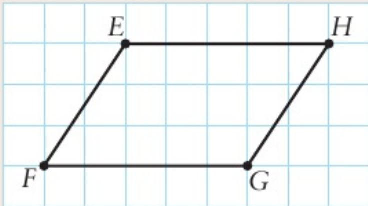
12 Calculate the area of the shapes in question 7.

13 Find the areas of the given shapes drawn to scale.

a

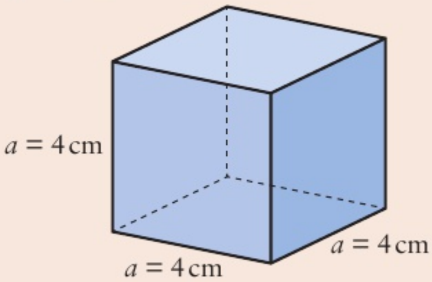


b

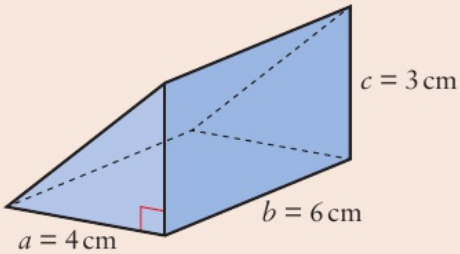


14 Calculate the surface area of the prisms shown.

a Cube



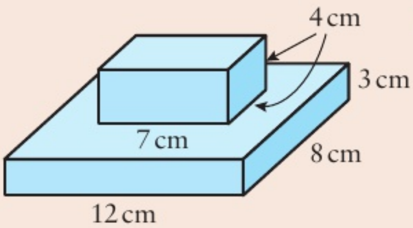
b Triangular prism



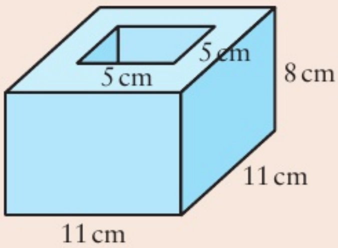
Challenge Q15

15 Calculate the surface area of the solids shown.

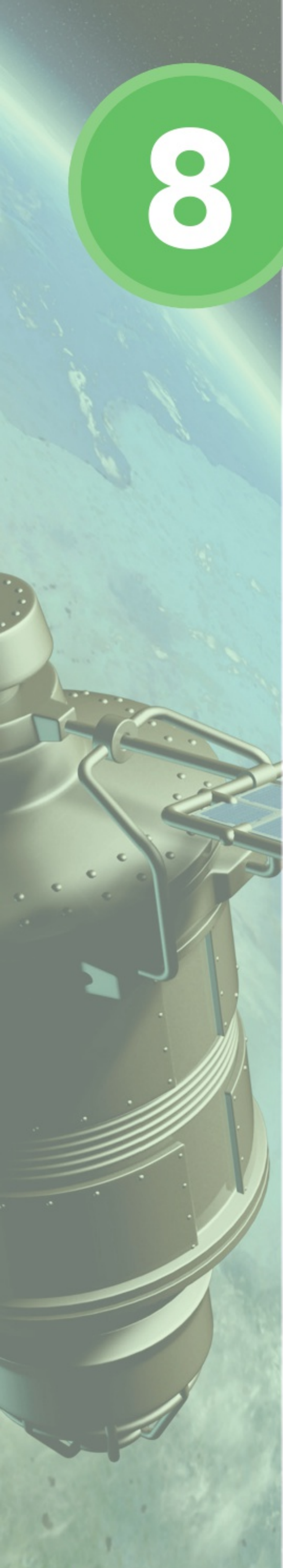
a



b



8 Loci and constructions



8 Loci and constructions

KEY CONCEPT

Form

RELATED CONCEPTS

Representation, Space, Systems, Validity

GLOBAL CONTEXT

Scientific and technical innovation

Statement of inquiry

The use of valid systems of shape construction allows us to represent changes of form in space in a technical and accurate way.

Factual

- What are ruler and compasses constructions?
- What is a locus of points?
- What does ‘equidistant’ mean?

Conceptual

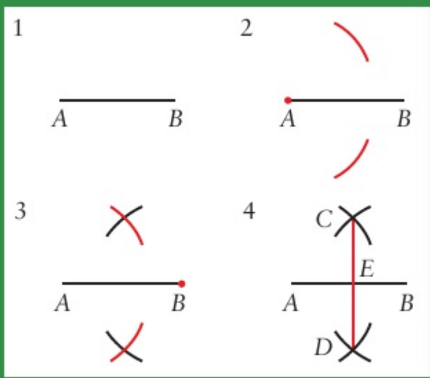
- What are the basic geometric elements of constructions using a ruler and compasses?
- What are ruler and compasses problems, and what does it mean to solve them?

Debatable

- Are constructions made using dynamic geometry software such as GeoGebra more precise than those made with a ruler and compasses?
- Can you construct any geometric figure or solve any geometric problem using a ruler and compasses only?

Do you recall?

1 You are given the following ruler and compasses construction:



Step 1 Draw segment AB.

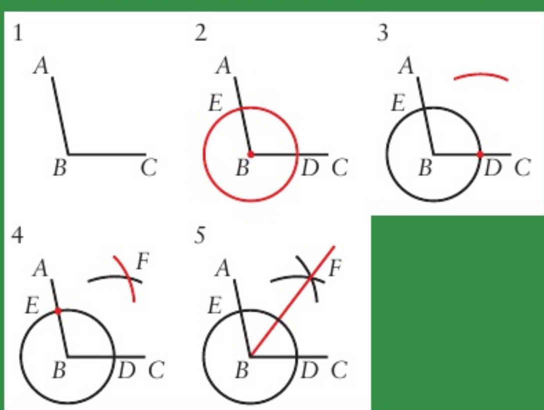
Step 2 With the point of the compasses at A and a radius more than half the length of AB, draw an arc above and below AB.

Step 3 With the point of the compasses at B and the same radius, draw arcs to cut the arcs already drawn at C and D.

Step 4 Join C to D.

Complete and justify: CD is ________ of segment AB.

2 You are given the following ruler and compass construction:



Step 1 Draw angle ABC.

Step 2 With the point of the compasses at the vertex B, draw an arc to cut both arms of the angle at D and E.

Step 3 Draw an arc from D.

Step 4 Draw an arc with the same radius from E to cut the arc drawn in step 3 at F.

Step 5 Join B to F.

Complete and justify: the ray BF is ________ of the angle ABC.

8.1 Introduction to ruler and compasses constructions

8.1.1 Constructing angles

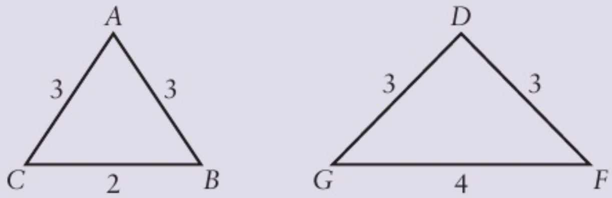
The study of geometric figures and their properties, and how to construct them, can lead to interesting discoveries of the world, as you will see in this chapter. At the same time, you will expand your understanding of mathematics as a universal language and learn to appreciate the powerful deductive reasoning that is the basis of any geometric construction.

Connections

Copying an angle using compasses and ruler only

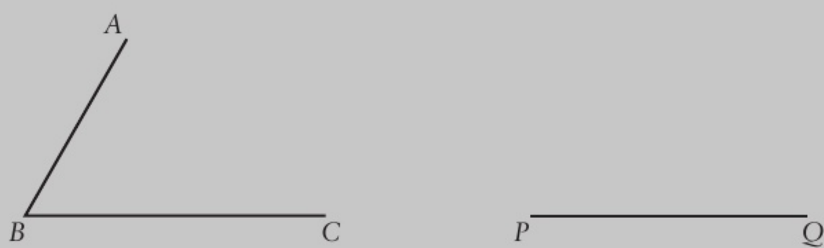
Explore 8.1

What can you say about $\sphericalangle CAB$ and $\sphericalangle GDF$ and $\sphericalangle ACB$ and $\sphericalangle DGF$?



Worked example 8.1

Given $\sphericalangle ABC$ and line segment PQ, copy $\sphericalangle ABC$ so that its vertex is at P.



Solution

Understand the problem

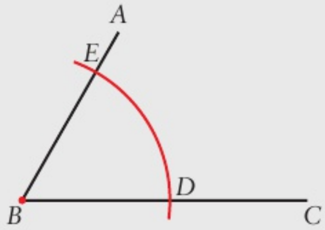
We need to construct an angle with one side PQ and with an equal measure to that of $\sphericalangle ABC$

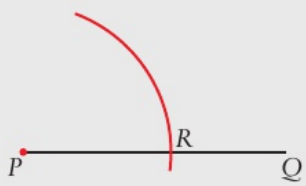
Make a plan

Since we can only use a straight edge and compasses, we use the compasses to measure the $\sphericalangle ABC$ and copy it to place the vertex at P. Then we can use the straight edge to draw the other side of the angle.

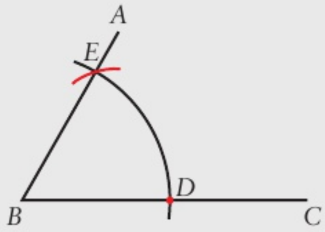
Carry out the plan

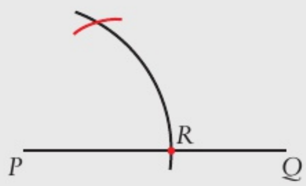
Step 1. Set the point of the compasses at B and draw an arc that intersects line segment BC at D and BA at E. Using the same radius, draw an arc with centre P that intersects line segment PQ at R.



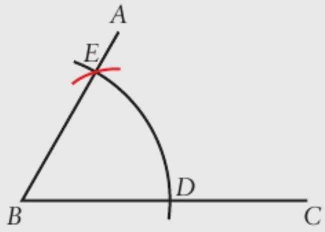


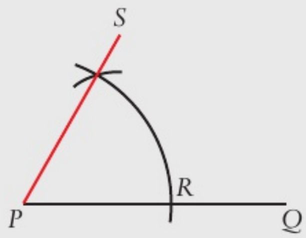
Step 2. Open the compasses to distance DE. Draw an arc with this radius and centre R that intersects the arc of Step 1. Here the arcs drawn from D and R have radius DE.





Step 3. Use the ruler to draw the line segment that passes through P and the intersection of the two arcs. Call it PS. Then, $\sphericalangle ABC$ is congruent to $\sphericalangle SPQ$.





Look back

Use a protractor to check the result of Step 3.

Reflect

Why are $\sphericalangle ABC$ and $\sphericalangle SPQ$ congruent?

Ruler and compasses constructions are based on mathematical principles, properties and theorems. Therefore, they are theoretical and rigorous, and they do not involve any kind of measurement. The only tools you can use in these constructions are a ruler and a pair of compasses; the ruler does not need markings – it is just a **straight edge**. Throughout this chapter, the term '*construction*' and the verb '*construct*' will mean ruler and compasses constructions.

Fact

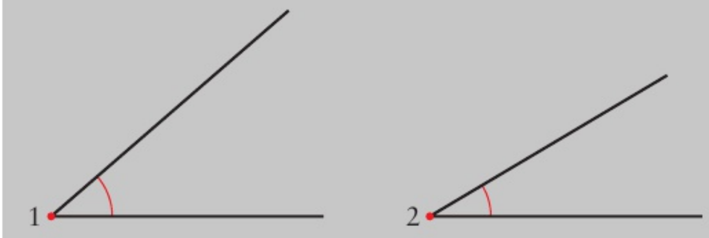
Very important: The arc with centre P and radius of length BE is called the **locus** of points that are at that same distance from P. A locus in geometry is a set of all points satisfying some specified properties or conditions. It can also be called a **locus of a point or object** where it is the path that point or object makes as it occupies all possible places defined by a set of rules. A locus can be a shape or an area. The plural of locus is **loci**.

Fact

The arc with this radius and centre R is the **locus** of points that are a distance equal to DE from R.

Worked example 8.2

Given angles 1 and 2, as shown, construct an angle with size equal to the sum of angles 1 and 2.



Solution

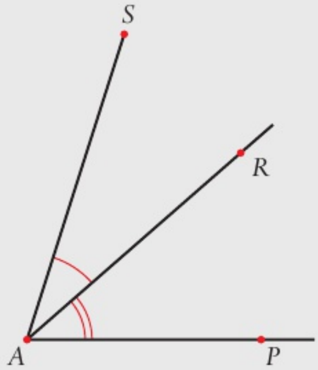
> **Connections**
>
> Adding angles

The key concept is addition of two angles. We need to construct two adjacent angles that share the same vertex and are congruent to ∢1 and ∢2.

Use the congruent angle construction twice, for ∢1 and ∢2, using a common line segment.

Draw a line segment *AP*. Using the construction shown in Worked example 8.1, construct an angle with side *AP* that is congruent to ∢1. Name it ∢*RAP*.

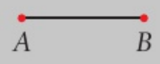
Using the same method, construct an adjacent angle to ∢*RAP* that is congruent to ∢2. Label it ∢*SAR*.



Looking back, the size of ∢*SAP* is the sum of the sizes of ∢*RAP* and ∢*SAR*, which are, respectively, congruent to ∢1 and ∢2. We can use a protractor to check that this is correct.

Worked example 8.3

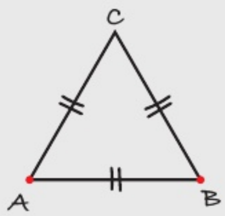
Given a line segment AB, construct an equilateral triangle whose sides are congruent to AB.



Solution

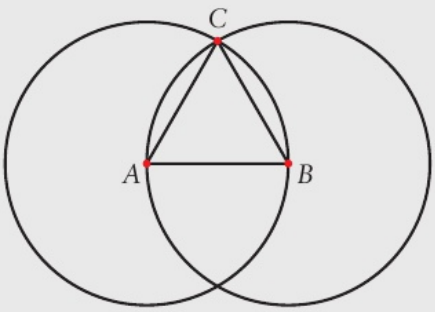
All three sides of an equilateral triangle are congruent. So, we must construct two more sides, say AC and BC, congruent to AB.

It is always a good idea to have a sketch of the figure to be constructed when possible. In this case the triangle would look like this:



Thus, we need to find the third vertex C such that AC and BC have a length equal to that of AB. So, we need to copy AB and draw circles with the same radius, AB.

Use a ruler to draw line segment AB. With A as its centre, draw a circle with radius AB. Point C must belong to this circle. Next, draw a circle with radius AB and centre B. Label the intersection of the two circles C. Connect C to A and to B.



We can check our construction by measuring the sides of the triangle. Are they congruent?

Hint

C lies on the locus of points at a distance equal to the length of AB from A and also on the locus of points at that same distance from B. So, it is on the intersection between these two **loci** (plural of locus).

Reflect

Is there another point satisfying these conditions?

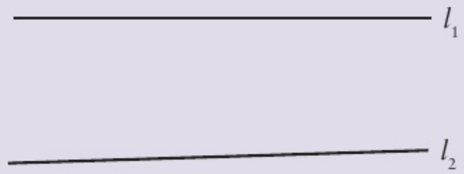
What are the sizes of the interior angles of triangle ABC?

8.1.2 Parallel and perpendicular lines

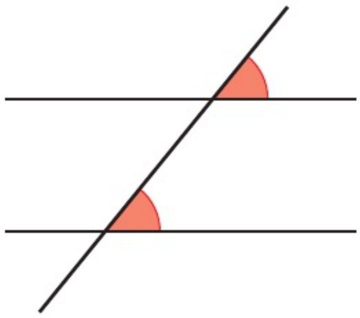
Parallel lines

Explore 8.2

Can you describe all the methods you know to determine whether the two lines below are parallel?



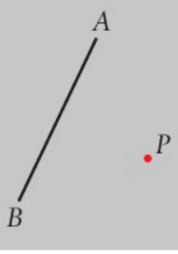
Recall that if we are given two parallel lines cut by a transversal, then the corresponding angles are congruent.



Applying this knowledge, we can construct lines that are parallel as shown in Worked example 8.4

Worked example 8.4

Given line AB and a point P not on AB, draw a line through P that is parallel to AB.



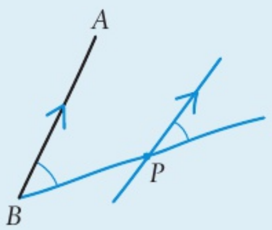
Solution

Start by drawing a rough sketch.

If we have a line through P that is parallel to AB, and we draw a transversal, then the corresponding angles will be congruent. Thus, our task is to draw a transversal.

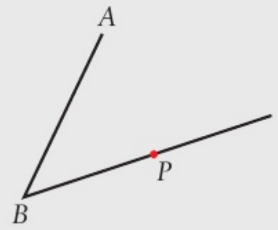
Hint

It is good practice to draw a sketch so that you are clear about what to look for.

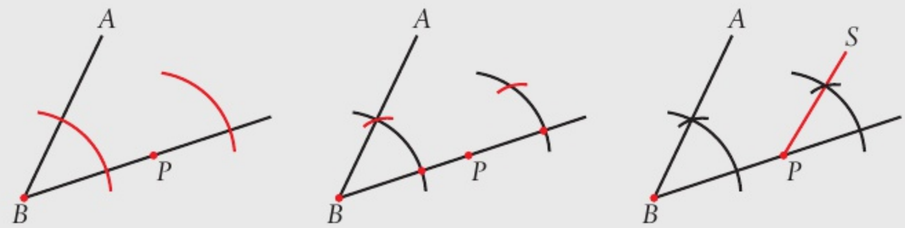


The transversal will meet the line AB at angle B. Copy the angle at B to the new position at P, making P the vertex of an angle congruent to angle B.

Step 1. Draw a line through B and P.



Step 2. As we did earlier, construct a new angle with vertex P congruent to the angle $\sphericalangle ABP$ by copying it.



Now, we have at P an angle congruent to the angle at B. Since they are corresponding angles, then the line SP is parallel to line AB.

Reflect

Are there any other lines through P that are parallel to AB?

Can you think of another method of construction?

Perpendicular lines

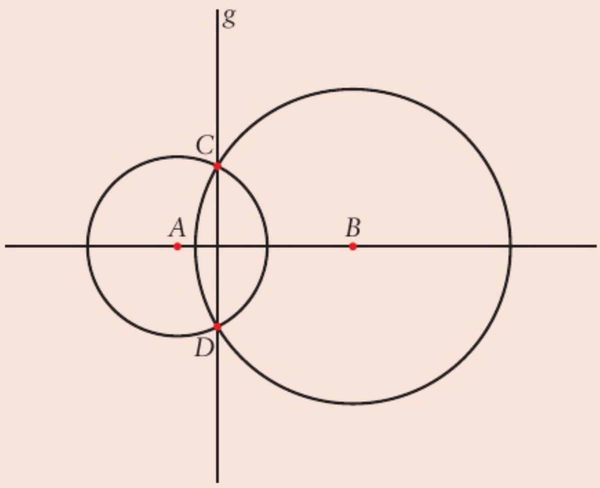
Two lines are perpendicular if they meet at a right angle.

Investigation 8.1

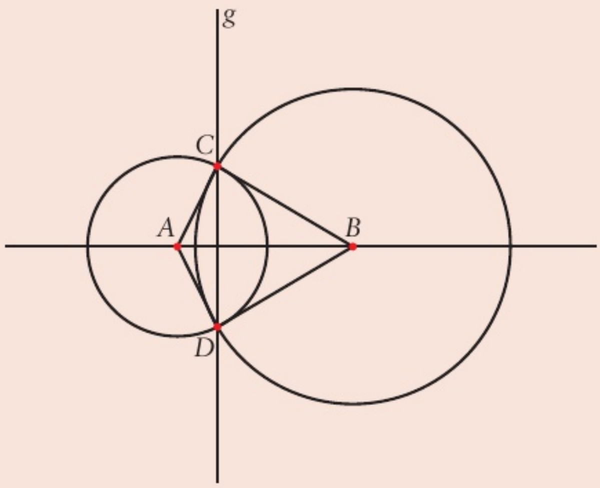
1 Start with a line segment AB and a point C that is not on it. Draw a line perpendicular to AB that passes through C. Call this line g.

2 Draw a circle with centre A and radius AC. Draw another circle with centre B and radius BC. Label the other point of intersection of the two circles D.

Draw the line segment CD. What do you notice about CD and g?



3 Now, connect the points A to C, C to B, B to D, and D to A. What shape have you drawn?



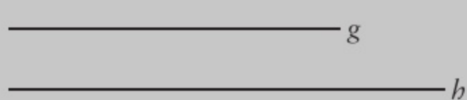
4 What are the properties of shape $ACBD$? Check that these satisfy those of the shape you named. Explain how you know.

5 What are line segments AB and CD of figure $ACBD$ called?

6 What geometric relationship is there between line segments AB and CD? Explain.

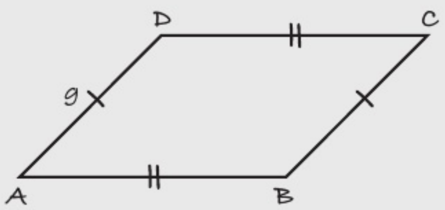
Worked example 8.5

Construct a parallelogram with the lengths of two of its sides equal to the lengths of the segments given below.



Solution

If we draw a sketch, we can be clearer about how we go about this construction.



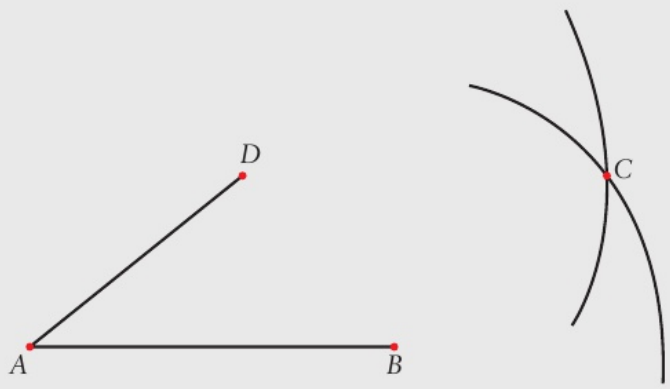
For a convex quadrilateral to be a parallelogram, opposite sides must be congruent.

No instructions about angles are given. Thus, there are no constraints there.

To satisfy the given condition we need to 'copy' the line segments as we have done earlier, which will give us three of the vertices. To find the fourth vertex, say C for example, we need to make sure that BC has the same length as g, and that DC has the same length as h.

As we did earlier, use a ruler and compasses to draw two sides AB and AD with a common endpoint A, and with measures g and h.

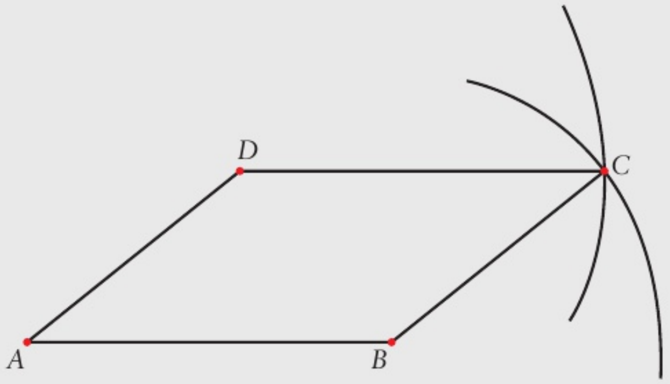
Now, using g as a radius, draw an arc with centre B. Then using h as a radius, draw an arc with centre D. Label the intersection of the arcs C.



Hint

Point C is on the locus of points a distance of g from B and the locus of points a distance of h from D. Hence, it is at the intersection.

Join D to C and B to C. $ABCD$ is a parallelogram



Looking back, check that the quadrilateral $ABCD$ satisfies the required properties:

Are $\overline{AD}$ and $\overline{BC}$ congruent?

Are $\overline{AB}$ and $\overline{DC}$ congruent?

Fact

Did you know that carpenters can check that lengths of wood are the same size using a big pair of compasses or dividers?

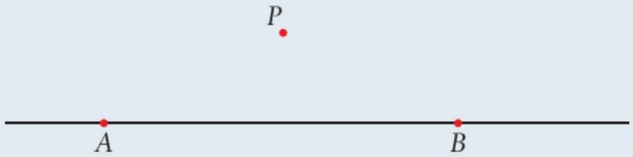
Reflect

Is this the only parallelogram we can construct?

Is there another way of drawing a parallelogram?

Practice questions 8.1

1 The diagram shows line AB and point P, not on the line.



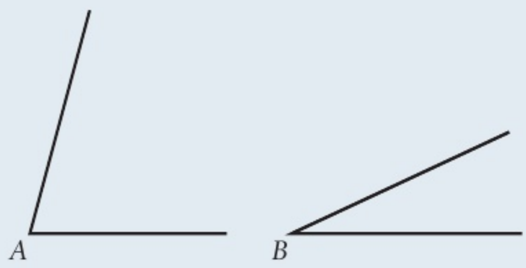
  a Construct a line segment that is congruent to segment AB. Explain all the steps.

  b Construct a line segment whose length is double the length of $\overline{AB}$. Explain all the steps.

  c Construct a line segment such that A is one endpoint and P is its midpoint.

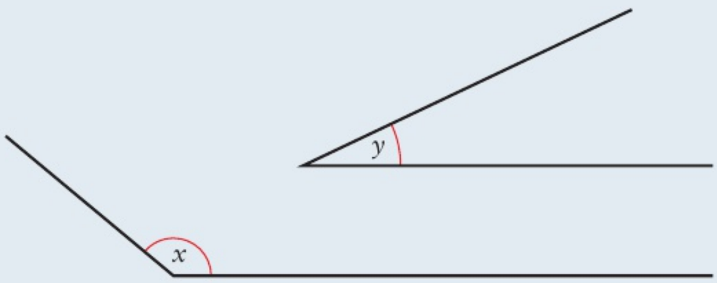
2 Draw any obtuse angle. Construct an angle congruent to it. Explain all the steps.

3 Draw angles A and B, where the size of $\sphericalangle A$ is larger than the size of $\sphericalangle B$, as shown below. Construct an angle C whose size is equal to the difference between the sizes of $\sphericalangle A$ and $\sphericalangle B$. Explain all the steps.

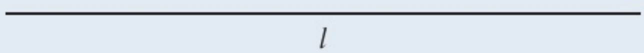


4 Draw angles with sizes x and y, as shown below. Then, construct angles with each of the following values.

a $2y$ b $x + 2y$ c $180 - x$



5 Construct an equilateral triangle with sides equal to l below. Then, use your protractor to measure each of the interior angles of the triangle.



6 Construct angles with each of the following sizes.

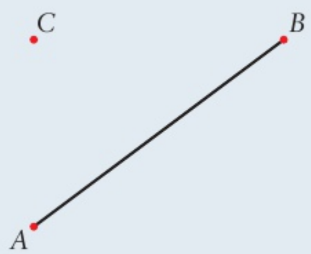
a 120° b 240° c 90° d 150°

7 Copy the diagrams below. In each diagram, construct:

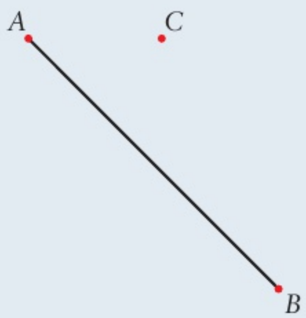
i a line parallel to AB going through C

ii a line perpendicular to AB going through C.

a

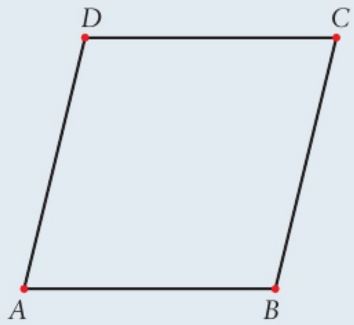


b

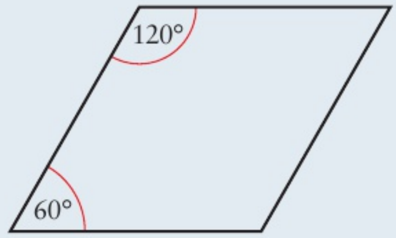


8 Construct the following figures.

a A parallelogram congruent to the one below.

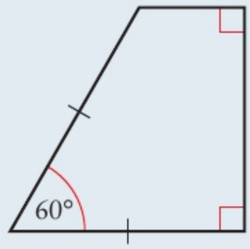


b A rhombus with angles as shown.



Challenge Q8c

c A trapezium with the angles and sides as shown.



d A square with side length of 4 cm.

Challenge Q9

9 Describe a method for constructing parallel lines that uses alternate interior angles. Then draw a line l and a point A not on l. Use your method to construct a line parallel to l that passes through A.

8.2 Loci

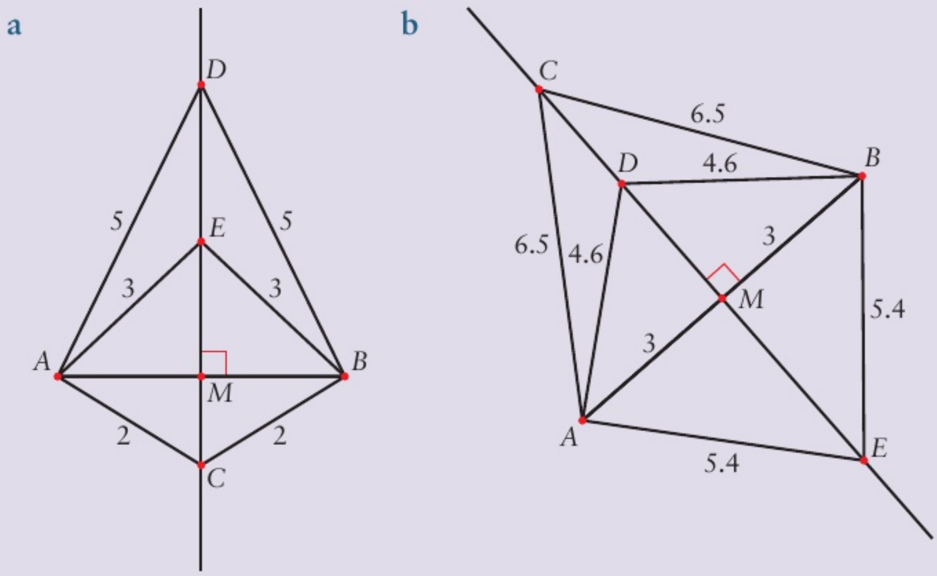
8.2.1. Bisectors

Reminder

The **perpendicular bisector of a segment *AB*** is a line perpendicular to *AB* that passes through the midpoint of segment *AB*.

Explore 8.3

In each of the following diagrams, you are given a few points that lie on the perpendicular bisector of a line segment *AB*, along with their distances from the endpoints of the segments.



What do you observe?

Write down a statement that describes a general property that all the points on a perpendicular bisector satisfy.

Can you justify why the property you discovered is true?

Explain why all the points on a perpendicular bisector satisfy the general property you found.

Reflect

How many perpendicular bisectors does a line segment have?

A perpendicular bisector is an example of a locus of points. A perpendicular bisector of a line segment is the locus of all the points that are equidistant from the endpoints of the segment.

Fact

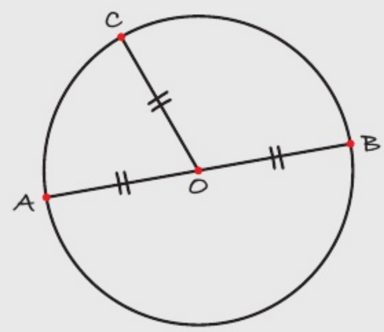
Equidistant means the same (equal) distance.

Worked example 8.6

Draw a circle that passes through the points A, B and C.

Solution

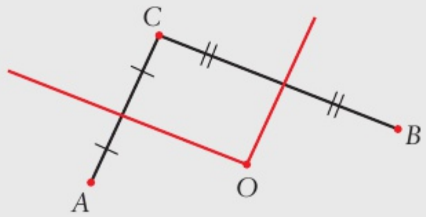
Draw a sketch of the situation.
It does not have to be accurate.



We are required to draw a circle. Thus, we need to find its centre and radius. Since the three points are on the circle, they are at equal distance from the centre. That is, the centre is equidistant from A, B and C.

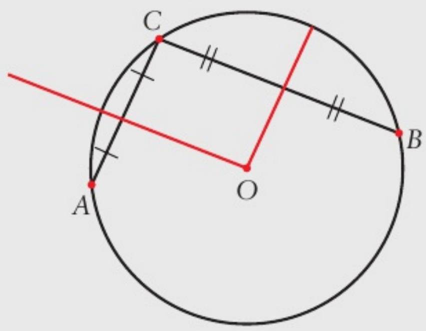
Let the centre be O. If O is equidistant from A and C, then it lies on the perpendicular bisector of segment AC. Similarly, it lies on the perpendicular bisector of segment BC.
So, O is on the intersection of these two loci.

We construct the perpendicular bisector of each of the segments above using what we know from earlier discussion.



Now, O is equidistant from C and A, and it is equidistant from C and B. Therefore it is equidistant from A, B and C. So, O is the centre of the circle.

Now, with O as centre and a radius equal to the distance from O to any of the points A, B or C, we draw a circle.



Looking back, we can check our work by considering the intersection of the perpendicular bisector of segment AB with the other two. In fact, it passes through O.

Hint

The circle is called the **circumcircle** of triangle ABC. Its centre, O, is the **circumcentre** of the triangle.

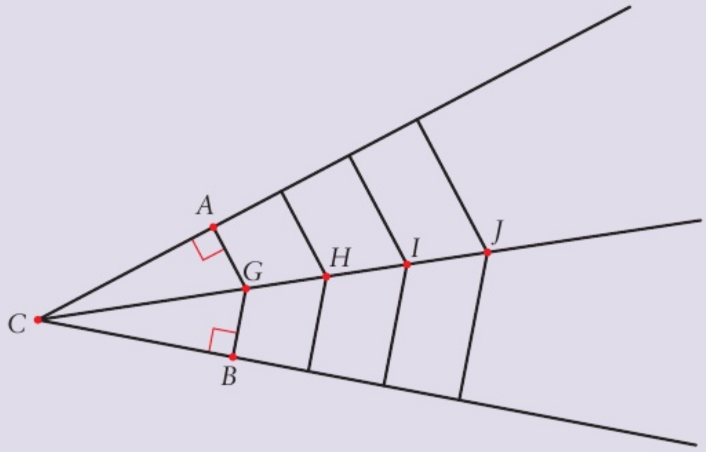
Reflect

Does the construction above work if the three points are collinear?

Is the circle found the only one possible?

Explore 8.4

Draw an angle ACB and its angle bisector, as shown below. Take several points on the angle bisector, (G, H, I and J) and measure with available tools the distance from each point to both sides of the angle.



What do you observe about these distances? Use your findings to write a definition of angle bisector.

Fact

The distribution of the veins of leaves often follows a very curious pattern. The primary vein resembles the angle bisector of angles whose sides are all parallel lines. This pattern is then repeated for the secondary veins, and so on. The veins transport water, minerals and food throughout the plant. The angle bisector structure ensures that the nutrients are transported in the most efficient way throughout the leaf.

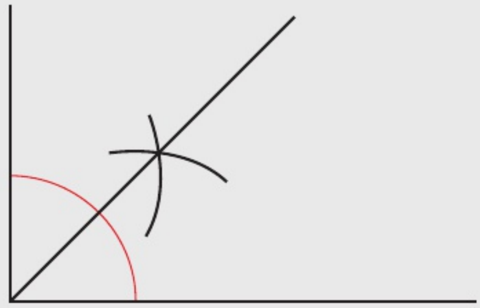
Worked example 8.7

Construct an angle of 45°

Solution

An angle of 45° is half of a right angle. Therefore, the problem asks you to bisect a 90° angle.

First, construct a right angle using the perpendicular bisector construction. Then apply the angle bisector construction to your diagram.



Check that the angle is 45° using a protractor.

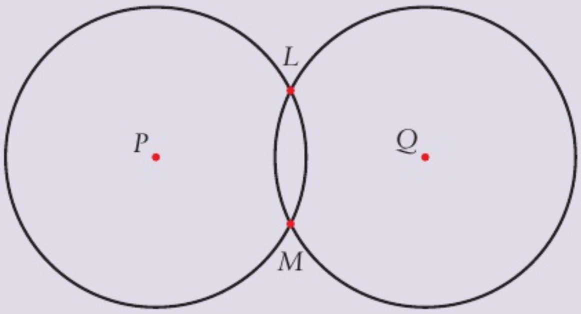
Reflect

With what you have done in mind, what other angles can you construct?

8.2.2 Circles and other loci of points

Explore 8.5

The diagram shows two points P and Q. The circle on the left shows the locus of all points a certain distance r from P. The circle on the right shows all the points the same distance r from Q.



a What can you say about the points L and M ?

b Copy the diagram on a piece of paper and do the same construction with different sized circles with centres P and Q (different values of r).

What do you notice about the set of points like L and M?

c Draw the line PQ. What is the relationship between the line PQ and the set of points discovered in part b?

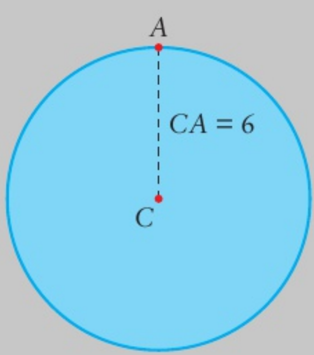
Hint b

You can use available technology to do the work.

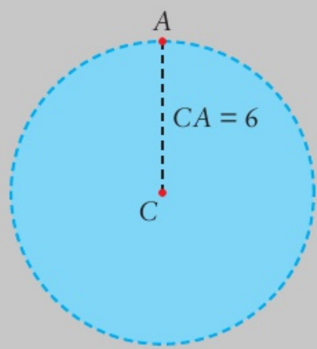
Worked example 8.8

Describe each of these shaded areas as a locus of points.

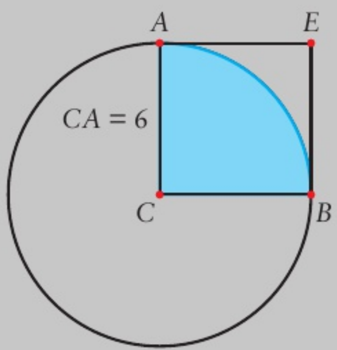
a



b



c



Solution

Describing the figures as a locus means identifying the conditions that all points in the shaded areas satisfy. To do this, we need to interpret the diagrams correctly.

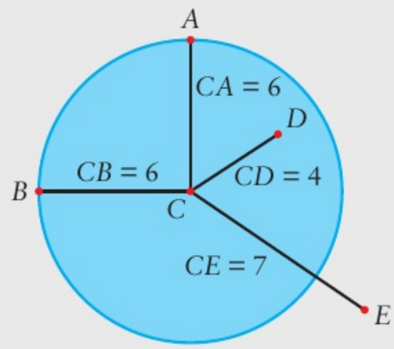
In diagram a the boundary of the circle is solid. This means that both the interior of the circle and the boundary are included in the locus.

In diagram b the boundary of the circle is dashed. This means that the interior of the circle is included in the locus but the boundary is not.

In diagram c the shaded area belongs to both the area of the circle **and** the square, that is, the shaded area is the intersection of the circle and the square.

Take an arbitrary shaded point in diagrams a, b and c. Identify the conditions that it satisfies. Repeat for other points, including some on the boundary and some that are not in the shaded region. Check that the conditions are satisfied by all the shaded points and are not satisfied for the unshaded ones. Write a statement that summarises the findings.

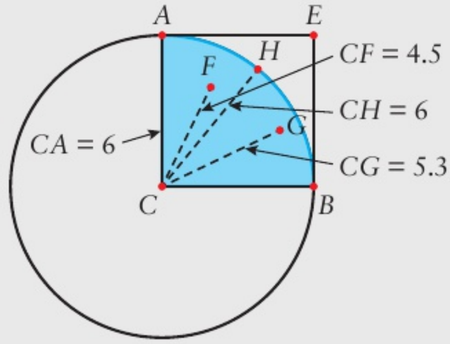
a In diagram a, all the shaded points are at a distance less than (the interior points) or equal to (the points on the boundary) 6 cm from C. So, diagram a is the locus of all the points that are at a distance less than or equal to 6 cm from C.



b In diagram b, all the shaded points are at a distance strictly less than 6 cm from C.

So, diagram b is the locus of all the points, and only those, that are at a distance of less than 6 cm from C.

c In diagram c, all the shaded points are at a distance less than (the interior points) or equal to (on the boundary) 6 cm from C **and** are in the square $CAEB$. So, diagram c is the locus of all the points, and only those, that are at a distance of less or equal to 6 cm from C **and** are inside the square.



Looking back, we have written three statements that describe the loci of points represented in these diagrams. We can check that they are true for other points in the shaded regions, so that we can conclude that all points satisfy the conditions described.

Practice questions 8.2

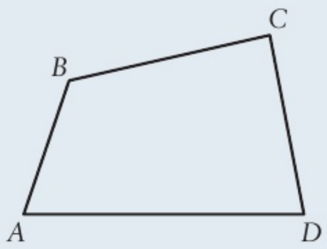
1 Draw two line segments similar to those shown below and construct the perpendicular bisector of each.

a 

b

2 Copy quadrilateral *ABCD* into your notebook.

a Construct the midpoint of each side.



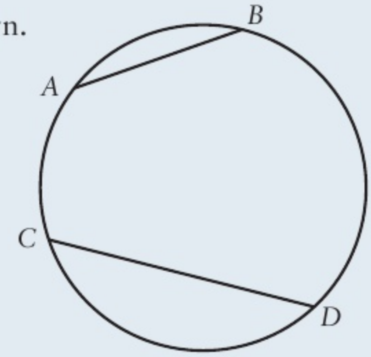
b What shape do you obtain by joining the midpoints?
Write a general statement that describes your result.

c Check your statement on two more examples, including a concave quadrilateral.

3 Draw a circle and any two chords as shown. Label the chords $\overline{AB}$ and $\overline{CD}$.

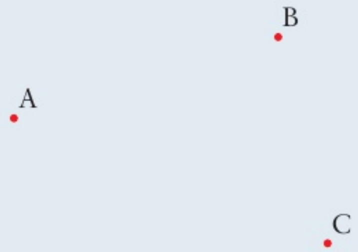
Construct the perpendicular bisectors of $\overline{AB}$ and $\overline{CD}$ and label the point of intersection of the perpendicular bisectors *E*.

Explain what you notice.



4 a Draw three points arranged like *A*, *B* and *C* below. Use ruler and compasses constructions to locate a point that is equidistant from *A*, *B* and *C*. Explain the steps of your method.

b Draw a circle that passes through points *A*, *B* and *C*.



c Mark three points in any positions. Is it always possible to draw a circle that passes through all of them? In which situation is it not possible to draw a circle?

Fact Q5a

The angle bisectors will meet at a point called the **incentre**.

Hint Q5b

Use the method you know for constructing perpendicular lines.

Fact Q5c

The incircle is the largest circle that touches each of the three sides of the triangle at exactly one point from the inside (i.e. it is **tangential** to the sides).

5 Draw any triangle ABC in your notebook.

a Construct the angle bisectors of each of the three interior angles of the triangle. They pass through one point called the **incentre**.

b Construct three perpendicular line segments from the incentre to each side of the triangle.

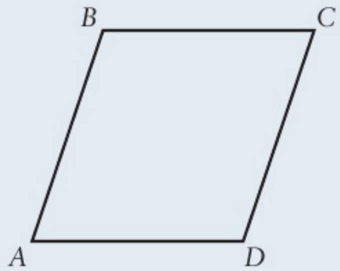
c Label the points where the perpendicular line segments meet the sides of the triangle as D, E and F. Justify that we can use the incentre to construct a circle (called the incircle) that passes through D, E and F. What do you notice about the incircle?

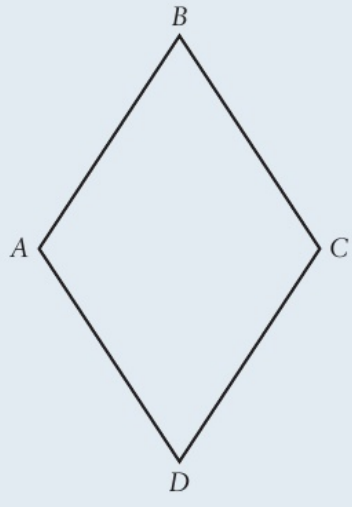
6 Draw a parallelogram, a rhombus, a rectangle and a square similar to those shown below. Using a ruler and compasses, bisect angle ABC in each shape.

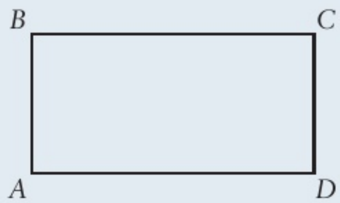
a In which shape does the angle bisector also bisect the opposite angle?

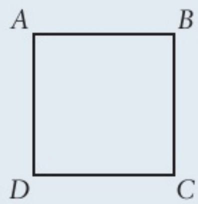
b How are the angle bisectors of two opposite angles in a parallelogram related?

c In which shape do the diagonals bisect both of the angles through which they pass?









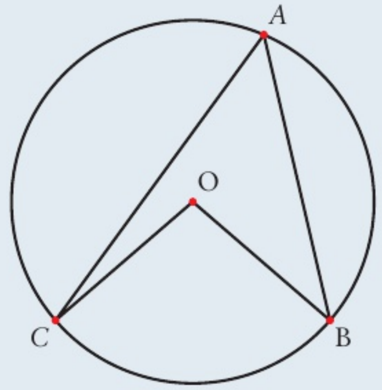
7 Draw a circle with centre O. Create angle CAB (the angle at the circumference), using points A, B and C on the circumference and angle COB (the angle at the centre), as shown below.

a Use a ruler and compasses to bisect angle COB.

b Use a protractor to measure angle CAB and one of the angles obtained by the bisection.

c What do you notice?

d Move point A to a new position on the circumference but keep points B and C fixed. Measure the size of the new angle CAB. What do you notice?

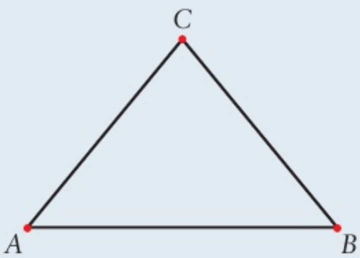


8 Draw an isosceles triangle ABC with base AB, as shown.

a Construct the perpendicular bisector of the base AB.

b Does this perpendicular bisector pass through C? Justify your answer.

c Construct the bisector of angle C. What do you notice?



Hint Q8b

Use the definition of a perpendicular bisector as a locus of points.

9 Several ruler and compasses constructions can be seen by paper folding. Work with a partner. Take turns being 'the convincer' or 'the skeptic' who must be convinced. When you are the convincer you have to be convincing! Give reasons for all your affirmations. Skeptics must be skeptical! Ask for reasons and justification that make sense to you. For each of the following questions, one student is the convincer and makes the shape while the other student is the skeptic. You should switch roles after each question.

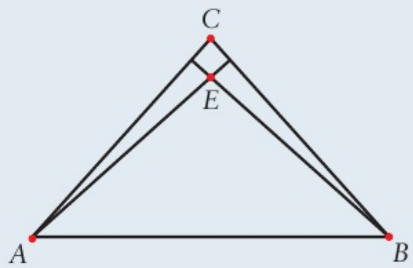
a Draw a line segment AB on a piece of paper. By folding the paper, construct the perpendicular bisector of AB.

b Draw angle ABC on a piece of paper. By folding the paper, construct the bisector of angle ACB.

c Draw a square on a piece of paper. By folding the paper, construct a square with exactly one quarter of the area of the original square.

d Draw a square on a piece of paper. By folding the paper, construct a triangle with exactly one quarter of the area of the original square.

10 Construct an isosceles triangle ABC with base AB, as shown below.

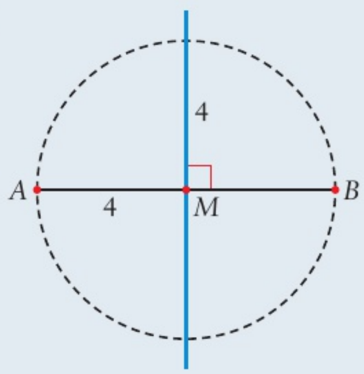


a Construct the perpendicular line segments from A to the opposite side BC, and from B to the opposite side AC. These are also called **heights** of ABC. Label the point of intersection of the perpendicular bisectors as E. Construct the perpendicular line segment from C to the opposite side AB. What do you notice?

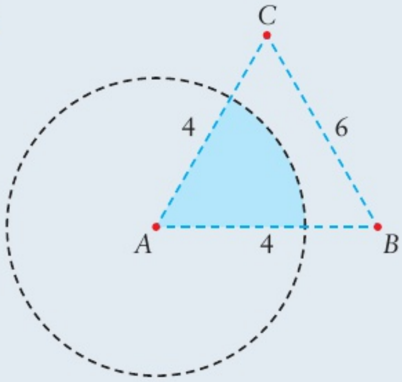
b Construct the incentre and the circumcentre of triangle ABC. What do you notice?

11 Describe the shaded region in each diagram as a locus of points. All measurements are in centimetres.

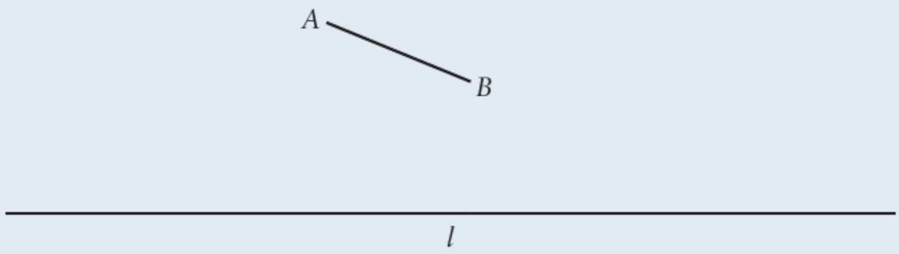
a



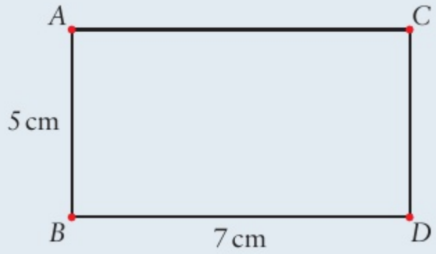
b



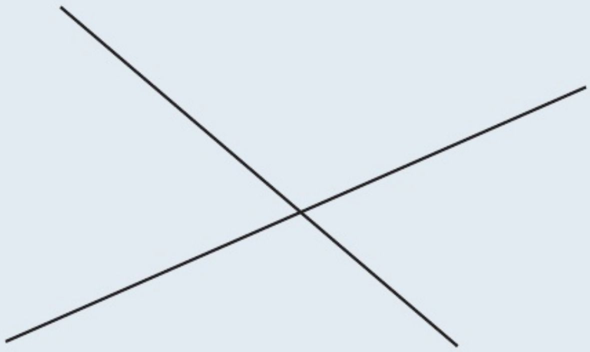
12 Given a line l and a line segment AB that is not parallel to l, as shown below, determine the point(s) equidistant from A and B that belong to l. Represent the situation on the diagram.



13 Construct a rectangle similar to the one below. Shade the locus of points in this rectangle that are less than or equal to 4 cm from A.



14 Find the locus of all the points that are equidistant from two non-parallel lines.



Hint Q14

Two non-parallel lines form 4 angles. Think about the definition of an angle bisector as a locus of points.

15 Find the locus of all points that are equidistant from two parallel lines.

Hint Q15

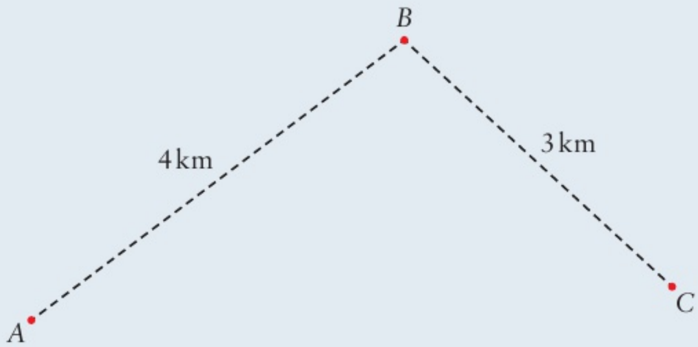
The distance from a point to a line is the length of the perpendicular line segment that joins the point to the line.

16 Given a line l, find the locus of all the points that have distance of 3 cm from l.

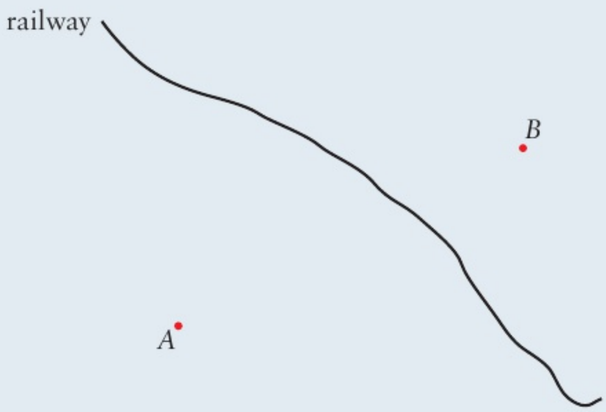
17 A new road is to be built between two cities, A and B. The two cities are 16 km apart. The new road is to be straight and equidistant from each town. Use a scale of 1 cm to 1 km to construct a scale diagram representing where the new road must be located.

18 The train station in a city is to be expanded and a new track must be built between an existing high-speed track and a low-speed track, which are 15 metres apart and parallel. The new track must be positioned so that its distance from the high-speed track is twice its distance from the low-speed track. Use a scale of 1 cm to 1 m to construct a scale diagram that shows where the new track must be located.

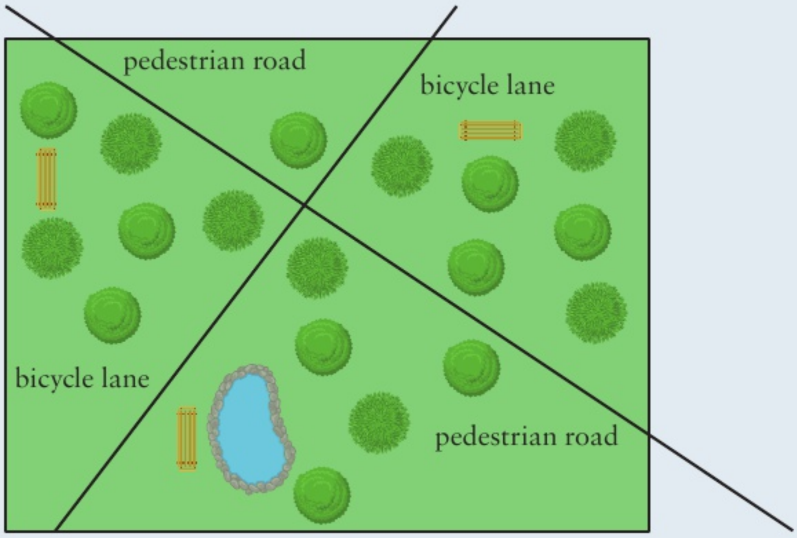
19 Three post offices, *A*, *B* and *C*, are located in a town, as shown below. *A* is 4 km away from *B*, and *B* is 3 km away from *C*. A fourth post office is to be built. The new post office will be positioned at an equal distance from the three existing post offices. Using a ruler and compasses construction, identify the location of the new post office. Use a scale of 1 cm to 1 km to construct a scale diagram.



20 Two towns are 10 km apart and a rail track passes between them, as shown below. A station is to be built along the railway track to serve the two towns. The station will be built at an equal distance from both towns. Using a scale of 1 cm to 1 km, construct a scale diagram to identify the position of the new station.



21 Adelina's dog got lost in a large fenced-in rectangular park. In the park, there is a bicycle lane and a pedestrian road that cross, as shown in the diagram. Adelina's dog is afraid of bicycles. Therefore, Adelina knows that she must look for her dog in the area closer to the pedestrian road than the bicycle lane. Copy the diagram and use a ruler and compasses construction to identify the area where Adelina's dog is likely to be found.



Challenge Q21

Self assessment

- Using ruler and compasses I can copy an angle.
- Using ruler and compasses I can construct parallel lines.
- Using ruler and compasses I can construct perpendicular lines.
- Using ruler and compasses I can construct parallelograms.
- Using ruler and compasses I can bisect an angle.
- Using ruler and compasses I can bisect a segment.
- Using ruler and compasses I can describe a perpendicular bisector as a locus of points.
- Using ruler and compasses I can describe an angle bisector as a locus of points.
- Using ruler and compasses I can describe a circle as a locus of points.
- Using ruler and compasses I can describe a composite figure as a locus of points.
- Using ruler and compasses I can solve problems using ruler and compasses constructions.
- Using ruler and compasses I can solve problems using loci.

Check your knowledge questions

1 Which of the following sentences are true?

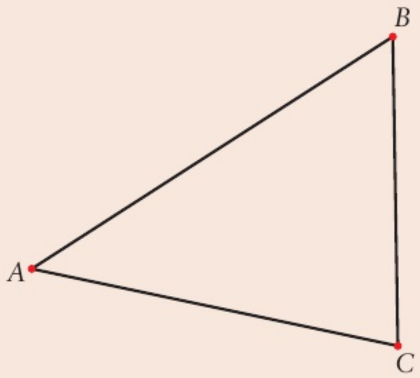
a Ruler and compasses constructions are used to measure geometric objects.

b Ruler and compasses constructions are based on theorems and properties and are rigorous processes.

c Ruler and compasses constructions can solve any geometric problem.

d Ruler and compasses constructions use ruler and compasses only, and do not involve any measurement.

2 In your notebook, draw a triangle ABC similar to the one below and use it for the constructions in parts a, b and c.



a Bisect angle *ABC*.

b Bisect line segment *AC*.

c Construct a perpendicular line to line *AC* going through *A*.

3 Construct an angle of 30°

4 Construct an angle of 135°

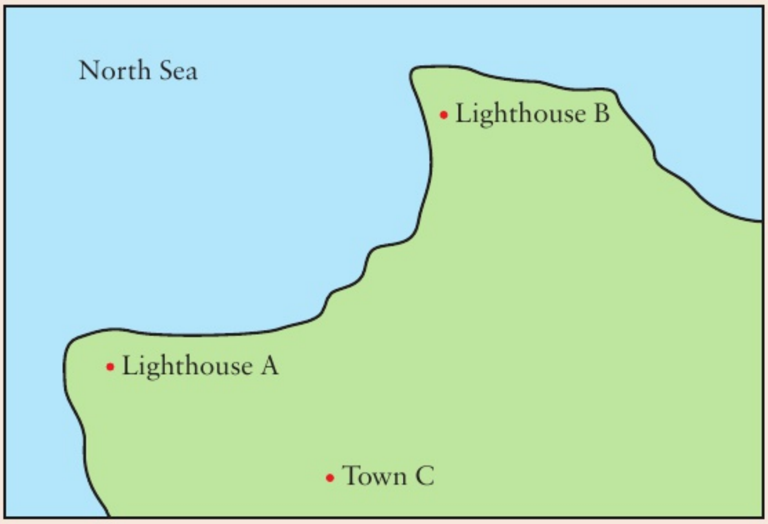
5 Draw any line segment *AB* and mark a point *C* on it. Use the angle bisector construction for the straight angle *ACB* as an alternative construction for drawing a line perpendicular to *AB*.

6 Construct a parallelogram where one of the angles measures 60°, and two sides measure 3 cm and 5 cm.

7 A square *ABCD* has sides of length 6 cm. Draw the locus of all points 4 cm from side *BC* of the square.

Challenge Q8

8 A boat has broken down in the North Sea. It is known that the boat was travelling on a line equidistant from both Lighthouse A and Lighthouse B, which are 10 km apart. It is also known that the boat was 15 km from Town C, which is 6 km from Lighthouse A and 8 km from Lighthouse B. Use a scale of 1 cm to 1 km to draw a similar map and locate the position of the boat.



9 Choose the correct answer. A point on the perpendicular bisector of a line segment AB is:

a equidistant from the endpoints of $\overline{AB}$

b equidistant from the endpoints of $\overline{AB}$ and the midpoint of $\overline{AB}$

a parallel to $\overline{AB}$

d perpendicular to $\overline{AB}$.

10 The bisector of an angle is the locus of all the points, and only those:

a that are equidistant from the vertex of the angle

b such that the sum of the distances from the two arms of the angle is constant

c that are equidistant from both the arms of the angle.

Choose the correct answer.

11 Two towns A and B are 30 km apart. The police station at town A responds to all calls within a radius of 25 km. The police station at town B responds to all calls within a 15 km radius. Use a scale diagram to shade the area that is served by both police stations.

12 Follow the steps below.

- **a** Draw a line segment AC.
- **b** Construct the perpendicular bisector of $\overline{AC}$ and mark the midpoint as M.
- **c** Draw a circle with centre A and radius AC.
- **d** Mark the two points of intersection of the perpendicular bisector and circle with centre A and radius AC, as D and B.
- **e** Join B with A and C, and D with A and C

  What kind of polygon is $ABCD$?

9 Statistics

9 Statistics

KEY CONCEPT

Form

RELATED CONCEPTS

Representation, Patterns, Validity

GLOBAL CONTEXT

Fairness and development

Statement of inquiry

Representing data in different forms allows us to find patterns so that we can make fair, valid and informed decisions.

Factual

- What are primary and secondary data sources?
- How can we calculate the mean for grouped data?

Conceptual

- How can we make sure that the data we collect is fair?
- Are grouped data and continuous data the same?
- Why is it important to represent data in different forms?

Debatable

- Can data be used to misinform as well as inform?
  How can we tell the difference?

Do you recall?

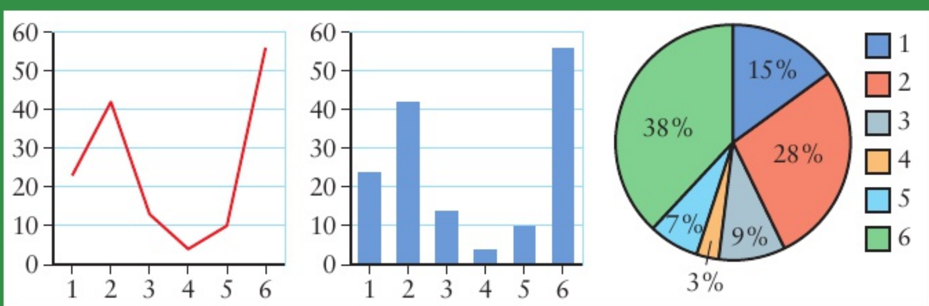
1 What is discrete data and what is continuous data?
For the list of variables below, classify them as discrete data or continuous data:

- **a** Shoe size
- **b** Length of your foot
- **c** Number of bananas
- **d** Mass of a bunch of bananas
- **e** The number of times you go to the store in a week
- **f** The time you spend in the store

2 How do you calculate the mean and identify the median and mode for discrete data?
For the lists of numbers below, state the mode, identify the median and calculate the mean.

- **a** 23 42 13 4 12 3 31 6 6
- **b** 11 13 15 18 19 18 21 21

3 What types of chart can you use to represent data?
Each diagram below represents a set of data. Name each type of diagram.



4 A list of discrete data can be represented as a stem-and-leaf plot.
What is meant by the stem, and what is meant by the leaf?
For the list of numbers below, state the stem and state the leaf for each number:

24 57 129 12.4 2345 2.38

Represent the following data set in a stem-and-leaf plot:

9	27	32	11	29	36	14	35	3	22
9	8	4	8	7	15	21	10	10	19
30	39	36	16	22	24	36	23	12	3

Fact

Statistics is the science of collecting, classifying, describing, representing and analysing data so that fair and valid decisions can be made.

Hint

You can explore the stem-and-leaf plot by using the Gizmo here:

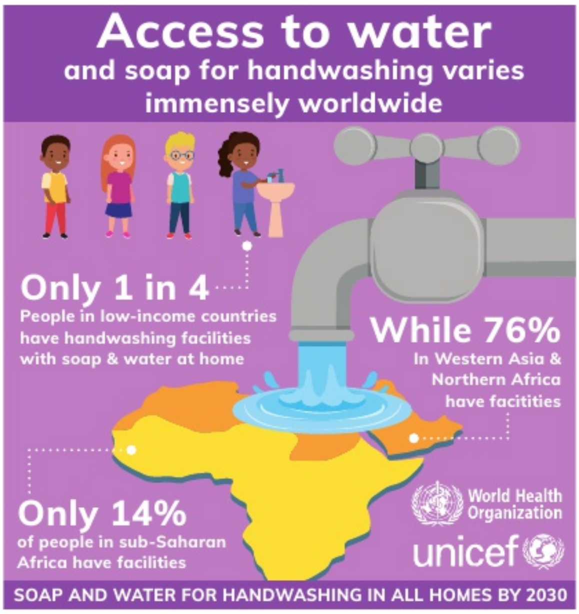
Enter your data values onto the line plot and watch the stem-and-leaf plot appear!

9.1 Collecting data

Fact

The WHO is the World Health Organization.

The data contained in the WHO's infographic 'Access to water' shows how statistics can be used to highlight differences in people's access to key resources.



Explore 9.1

Dana lives in a city in the UK and she feels that there should be many more trees planted around her school. She would like to present her principal, Mr Smith, with data to support her suggestion for part of the sports field to be developed into a place with trees and benches. She also feels that this would be a great addition to the community as there are only a few parks in the area around her school.

We Need Trees!

Trees are more important than Sport

I think we need more trees around school.
We need places to eat outside and we need to have lots of shade. Please answer these questions so I can convince Mr. Smith to plant trees on the sports field.

Trees are very important, aren't they?

☐ Agree ☐ Disagree

How many times do you use the sports field?

☐ Never ☐ Sometimes

Would you like to:

Be able to sit and chat with friends? ☐ Yes | ☐ No

Save the planet? ☐ Yes | ☐ No

Dana collects data in two ways:

1 She conducts a survey of her close friends, using the 'We Need Trees' survey.

2 She researches online and downloads data about the benefits of shaded outdoor seating areas in southern Italy. The data was published by a tree planting company.

Can you examine the reliability of the data collection methods Dana is using?

Social skills

Thinking skills

Hint

When carrying out a survey it is important to make sure that you collect information that is both relevant and unbiased.

The data collected should be fair so that it informs your decisions.

Data collection should not be designed to confirm your own opinions; rather, it should be collected to either prove or disprove a hypothesis.

When collecting data, you can choose to generate your own data, for example by conducting an experiment or by using a survey. This is called **primary data** because you are the primary user of the data.

You can also choose to use data collected by someone else. This is called **secondary data** because you are the secondary user of the data.

9.1.1 Primary and secondary data

Look again at Dana's survey questions in Explore 9.1.

We Need Trees!

Trees are more important than Sport

I think we need more trees around school.
We need places to eat outside and we need to have lots of shade. Please answer these questions so I can convince Mr. Smith to plant trees on the sports field.

Trees are very important, aren't they?

☐ Agree ☐ Disagree

How many times do you use the sports field?

☐ Never ☐ Sometimes

Would you like to:

Be able to sit and chat with friends? ☐ Yes ☐ No

Save the planet? ☐ Yes ☐ No

In the first question, she has introduced bias because she is giving her opinion. The people who she asks may feel obliged to agree with her. This type of question is called a leading question.

The second question is too vague, as the answer options could mean different things to different people.

The third question is very unlikely to generate any 'No' answers, because who wouldn't want to do these two things!

When designing questions for a survey, you must think carefully about what you want to find out and what data you need.

It is best to ask short and simple questions with answer choices that will lead to the information you need.

Worked example 9.1

The Carbon Trust is an independent organisation that advises governments on improving carbon emissions and how to develop renewable energy sources. They have been investigating the use of large wind farms off the coast of North Wales in the UK. All of their data supports the development of these farms as an alternative energy source. However, they would like to understand the impact these would have on the residents of North Wales. They have designed a survey to gather information.

Decide whether or not the following questions would be suitable for their survey. If not, give a reason and rewrite the question to improve it.

Qu 1

What age are you?

☐ less than 18 ☐ 18 to 30 ☐ 30 to 80

Qu 2

What do you think of wind farms?

....

Qu 3

Do you agree that wind farms spoil the view?

☐ Yes ☐ No

Solution

Understand the problem

The questions in a survey must be relevant, they must not introduce bias and they must allow us to generate the data that we need.

Make a plan

By critically evaluating the questions we can determine if they are fair and relevant. If they are not, we can suggest alternatives that are more suitable.

Carry out the plan

Qu 1 is a fair and relevant question because it is trying to make sure that all age groups are consulted. However, the questions need to be changed because a person who is 30 has a choice of box to tick and someone who is over 80 doesn't have a box to tick.

An improved question could be:

What is your age range?

☐ 18 and below ☐ 19 to 30

☐ 31 to 60 ☐ over 60

Qu 2 is an open question, and although it might give information that is useful for the company, it is vague and difficult to analyse because each response will be different. If the survey is to determine the level of interest in wind farms as a renewable energy source, an alternative question could be:

To what extent do you agree or disagree with these statements?

I understand that wind farms produce energy.

☐ strongly agree ☐ agree

☐ disagree ☐ strongly disagree

☐ don't know

Wind farms should be built in Wales as alternative sources of energy.

☐ strongly agree ☐ agree

☐ disagree ☐ strongly disagree

☐ don't know

Qu 3 is a leading question, as the reader is being drawn into agreeing with the question.

An alternative question could be:

Would a wind farm built in your area have an impact on your environment?

☐ yes ☐ no ☐ don't know

If yes, which of the following do you feel would be impacted most?

☐ the view ☐ the noise level ☐ other

If other, please state:

....

Look back

The original questions were unfair. They either excluded people, were leading people to give a particular response, or were too vague to generate relevant data. The alternative questions are fairer as they allow people the opportunity to give their own opinion, and the data generated is more easily evaluated.

Reflect

What other questions could The Carbon Trust have used in their survey? Discuss your ideas with your group.

Look again at Dana's choice of secondary data in Explore 9.1. The choice of secondary data meant that she was using information that was not relevant to her own situation. Would it have made a difference to her claim if she had sourced information specifically related to the UK? Would it have made a difference to her claim if she had sourced information that was not from a tree planting company?

Investigation 9.1

Search online to find the information required to answer the following questions.

1. Find a list of countries that are members of the European Union.
2. Who presented data in 'polar area diagrams' to investigate death rates in the Crimean War (1853–1856)?
3. Who invented the pie chart and what else did they invent?
4. Find data that identifies the country with the lowest percentage of the population with basic access to drinking water.

Fact

You can make internet searches more specific and source information from multiple perspectives by using search commands.

- Country codes can be used to find the perspective of a particular nation. For example, typing **rugby "site:nz"** into your search engine (including the quotation marks) will give just New Zealand websites about rugby.
- Including 'ac' will yield verified academic institution sources; for example typing in **rugby "site:ac.nz"** will give New Zealand academic sources for rugby.

Hint

When sourcing secondary data for a statistical survey you must check your source for relevance and verify its accuracy.

- Does it come from a reliable source? For example, is the website address a .gov or an .edu or is it another reliable and trusted organisation?
- Can the data be verified by another trusted source?
- Are you using sources with different perspectives?

Practice questions 9.1

1 Identify each of these data sources as primary or secondary.

  a Bryn records the times for his chemistry reaction and will use the data for his report.

  b Wendy is designing a website for her friend's new online business. She wants to find out people's opinions about layout and colours so she designs and distributes a survey.

  c Leroy was preparing for a UNICEF conference on children's access to education and used the data from their website to prepare his presentation.

2 The following questions are being considered for a survey by a school principal to gather information about the lunch options in the senior school. He thinks they are too vague. Rewrite them so that they are more focused.

  a At what time do you usually have lunch?

  b Do you have school lunch every day?

  c Do you like the lunches in school?

3 Mr Thorpe lives on a busy street and is becoming increasingly concerned by the volume of traffic. He decides to conduct a survey of his neighbours and send the results to the traffic department of his town. Rewrite his questions a and b so that they are not leading questions. Add some tick boxes to his question c with suitable categories.

  a The traffic along this street travels far too fast, do you agree?

     Yes   No

b I think there should be speed restrictions on our street.
Do you agree?

Definitely   Maybe

c How long have you lived on my street?

4 The spreadsheet below is a section of the Organisation for Economic Co-operation and Development's database on education in different world regions.

TABLE 5. EDUCATION

Countries and areas	Youth (15–24 years) literacy rate (%) 2011–2016*		Number per 100 population 2016		Pre-primary school participation: Gross enrolment ratio (%) 2011–2016*	
	male	female	mobile phones	internet users	male	female
SUMMARY						
East Asia and Pacific	99	97	109	52	77	77
Europe and Central Asia	–	–	125	74	76	75
Eastern Europe and Central Asia	100	99	129	64	60	59
Western Europe	–	–	122	83	97	97
Latin America and Caribbean	98	99	109	56	76	76
Middle East and North Africa	91	88	112	48	35	34
North America	–	–	123	78	72	70
South Asia	88	80	85	26	22	21
Sub-Saharan Africa	79	72	75	20	31	32
Eastern and Southern Africa	87	85	71	21	39	40
West and Central Africa	69	55	80	19	20	21
Least developed countries	80	73	68	16	23	24
World	**92**	**85**	**101**	**46**	**49**	**48**

a Use the data to answer these questions.

i Which region had the highest number of mobile phones per 100 in 2016?

ii Which region had the lowest number of internet users per 100 in 2016?

iii Which regions are missing from the OECD database on literacy rates for 15- to 24-year olds?

iv Which regions have a higher rate of female pre-primary school participation than male?

b Compare the literacy rates for the Middle East and North Africa region with the data available for the World.

5 Give an example of a database that you use, or that you know your school uses. What information does it contain?

6 Research what the General Data Protection Regulation (GDPR) is for EU countries. Do all countries have similar legislation?

Hint Q6

You may find this a useful start for your research:

Challenge

Research skills

Åsa was researching ideas for her Individuals and Societies project on trade, aid and development and needed to find information on the development indicators for Brazil and the USA.

She found these two databases:

1 The World Bank

2 The OECD

What information does Åsa need? Research development indicators to determine whether these would be suitable databases.

Are these organisations reliable sources? Justify your decision.

9.2 Organising and describing data

9.2.1 Organising data

Explore 9.2

The data below represents the level obtained by a group of Grade 9 students in an assessment. Organise the data into a form that enables you to see whether it was a fair test.

2	4	6	3	4	6	5	3	5	7
4	6	3	4	6	4	7	5	4	3
2	5	2	6	4	6	4	3	3	4
4	4	7	8	4	5	7	5	4	5
3	6	3	4	3	6	5	4	6	3
4	4	2	5	4	2	3	6	2	5
6	4	5	4	6	4	5	4	5	4
5	7	6	2	3	6	2	5	2	6
3	2	5	3	5	2	3	5	2	2
5	6	7	6	4	5	4	8	5	3

Hint

When data has been collected, there is usually a lot of information in a form that doesn't easily allow for patterns to be identified.

In order to see patterns, we need to organise the data and represent it in different forms.

9.2.2 Describing discrete data: mean, median, mode and range

Fact

The range is the difference between the highest and the lowest data value.

The mode is the most common data value; it has the highest frequency.

The median is the data value that lies in the middle of the list when all the data have been arranged in ascending numerical order.

The mean is the value that is obtained when the data have been evenly distributed across the number of data values. It is the average that is calculated by:

$$\text{mean} = \frac{\text{sum of all the data values}}{\text{the number of data values}}$$

Worked example 9.2

The data organised in the frequency table below represents the levels achieved by a group of students in an MYP assessment.

Level	1	2	3	4	5	6	7	8
Frequency	0	13	16	25	21	17	6	2

a State the mode.

b Calculate the mean.

c Calculate the range.

d Determine the median.

Solution

The data is given in a frequency table; therefore, the mode can be identified as the data value with the highest frequency value. The frequencies need to be considered when finding the middle value for the median; it is not simply the middle number of the levels. The mean also needs to be calculated using the frequency values; for example, the table states that there are 13 level 2 scores.

The mode can be identified from the table as the data value with the highest frequency.

The mean can be calculated by using the mean formula:

$$\text{mean} = \frac{\text{sum of (data value} \times \text{frequency)}}{\text{sum of the frequencies}}$$

The median is the data value in the middle of the observations when they are in numerical order, so the middle value needs to be found.

Fact

When data has a specific value, it is called discrete. It is data that can be counted. The number of people in a concert hall and how many books you own are both examples of discrete data.

When data is measured within a range, it is called continuous. The height of a cypress tree in metres or the mass of a shipping container in tonnes are both examples of continuous data.

Fact

A statistic is a measure that represents a data set.

Statistics that represent the centre value of your data are called **measures of central tendency**. The three most common measures of central tendency are the mean, the median and the mode. A statistic that represents the spread of the data is called a **measure of dispersion**. The range is a measure of dispersion.

a The mode is the value with the highest frequency, and this can be seen from the table.

The mode is 4.

b The mean is calculated by using the formula, and it is helpful to organise the calculations into a table first:

Level	Frequency	Level × Frequency	
1	0	1×0	0
2	13	2×13	26
3	16	3×16	48
4	25	4×25	100
5	21	5×21	105
6	17	6×17	102
7	6	7×6	42
8	2	8×2	16
Sum of the values	100		439

$$\begin{aligned} \text{mean} &= \frac{439}{100} \\ &= 4.39 \end{aligned}$$

c The range is calculated by:

$$\text{range} = \text{maximum value} - \text{minimum value} = 8 - 2 = 6$$

d The median is the data value in the middle when the data is in numerical order. This is more difficult with a frequency table than for a simple small data set. In this case we have 100 data values so our median value will be halfway between the 50th and the 51st data values. The position of the median value can also be found efficiently by using:

$$\begin{aligned} \text{position of the median value} &= \frac{\text{total frequency} + 1}{2} \\ &= \frac{100 + 1}{2} \\ &= 50.5 \end{aligned}$$

Therefore, the median is the 50.5th data value, which is the average of the 50th and 51st data values.

If we list the values in a table and accumulate the frequencies, we can identify the 50th and 51st data values. Accumulating the data gives the **cumulative frequency.**

Level	Frequency	Cumulative frequency	
1	0	0	
2	13	13	$0 + 13 = 13$
3	16	29	$13 + 16 = 29$
4	25	54	$29 + 25 = 54$
5	21	75	$54 + 21 = 75$
6	17	92	$75 + 17 = 92$
7	6	98	$92 + 6 = 98$
8	2	100	$98 + 2 = 100$

The median is the value in between the 50th and the 51st data values. From the cumulative frequencies, both the 50th and 51st data values are level 4.

So, the median is 4.

In this example, the mean is 4.39 and the mode and the median are both 4, so all of the measures of central tendency are approximately 4. The spread of the data is 6.

As the measures of central tendency are similar in value, the data is symmetrical. There are no values that seem out of place in the data set. The range value of 6 identifies a wide spread of the data. The mean, median and mode are similar, so this implies that the data is spread out evenly around the centre of the data.

The statistic values suggest that the test is fair.

Looking back, the data looks quite evenly spread out around the Level 4 score, the frequency of the maximum and the minimum values are small and the scores in the centre have the greatest frequencies. This agrees with the values calculated for the mean, median and mode.

Reflect

What could you conclude about the test if the mean was 7.5 and the range was 1?

What could you conclude if the mean was 2.4 and the range was 7?

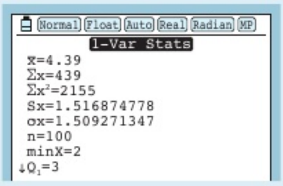
Fact

Cumulative frequency is the running total of the frequency values. To find the cumulative frequency of a data value, add the previous frequency values to the frequency of that data value.

Hint

You can use your GDC to perform calculations by using the one variable statistics (1-Var Stats) option.

A sample output



Practice questions 9.2.2

1 Determine the mean, median, mode and range for these sets of discrete data. Comment on the values you obtain. What do they tell you about your data?

a

x	1	56	45	89	65	90	67	52

b

x	1	56	45	89	65	90	67	52
f	15	5	3	4	7	1	2	5

2 The rainfall in a coastal town was measured in millimetres every week over 8 weeks during the summer season. The number of hours of sunshine was also recorded. The data are organised in the table below.

Rainfall in millimetres per week	1	18	4	2	0	2	23	3
Sunshine in hours per week	15	0	19	20	70	21	0	20

a State the mode for the rainfall data and the mode for the number of hours of sunshine for a week.

b Calculate the mean number of millimetres of rainfall and the mean number of hours of sunshine.

c Find the median rainfall and median number of hours of sunshine.

d Find the range of the rainfall and the hours of sunshine.

e If you were responsible for promoting the town to tourists, explain which measure of central tendency you would use in your advertising material.

f Would your answer to part e be fair to the tourists or to the town's population?

3 Monthly interest rates for Australia and USA for the period from December 2019 to August 2020 are given in the table below.

	Dec 2019	Jan 2020	Feb 2020	Mar 2020	Apr 2020	May 2020	Jun 2020	Jul 2020	Aug 2020
Australia	1.2	1.15	0.98	0.89	0.86	0.91	0.92	0.88	0.89
USA	1.86	1.76	1.50	0.87	0.66	0.67	0.73	0.62	0.65

a Calculate the mean interest rate for each country.

b Find the median interest rate for each country.

c Find the range of the interest rates for both countries.

d Using your values from parts a, b and c, compare the average monthly interest rates for these two countries over the given time period.

4 The yearly pharmaceutical sales data per capita for Mexico and Switzerland are given in the table below (in \$).

	2010	2011	2012	2013	2014	2015	2016	2017
Mexico	109.8	97.2	102.2	97.2	91.4	91.7	94.6	91.3
Switzerland	501.1	522.4	548.3	558.6	565.1	604.3	633.9	653.8

a Calculate the mean value for each country.

b Find the median value for each country.

c Find the range of the pharmaceutical sales data for Mexico and Switzerland.

d Using your values from parts a, b and c, compare the pharmaceutical sales data per capita for each country over the period from 2010 to 2017.

Fact Q4

per capita means per person

5 The average annual salary data for Sweden and Latvia (converted to \$US) are given in the table below.

	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019
Sweden	41500	42180	43133	43647	44240	44849	45552	45818	46062	46695
Latvia	18982	18316	19067	20005	21307	23258	24663	25620	26765	28454

a Calculate the mean average yearly salary for the period for both countries.

b Find the median average yearly salary for the period for both countries.

c Find the range for the salary data for both countries.

d Using your values from a, b and c, compare the average yearly salaries for each country over the period 2010 to 2019.

6 Mr Brubaker was reviewing the IB diploma scores for his English classes over time. The results for his 2017 classes are given in the table:

Score	1	2	3	4	5	6	7
Number of students	1	3	8	12	16	10	5

a State the mode IB score for 2017.

b Determine the median score for 2017.

c Calculate the mean score for 2017.

d Calculate the range for the 2017 data

7 The results for Mr Brubaker's 2020 classes are given in the table:

Score	1	2	3	4	5	6	7
Number of students	0	1	2	13	16	22	1

a State the mode IB score for 2020.

b Determine the median score for 2020.

c Calculate the mean score for 2020.

d Calculate the range for the 2020 data.

e Compare the data for Mr Brubaker's 2017 classes and his 2020 classes.

Is there a reason why the 2020 data could be collected differently? Is it fair to compare the two data sets if the data is collected differently?

9.2.3 Describing grouped data: mean, median and mode

If there are a lot of data values in a data set it is more efficient to organise the data into groups to be able to identify patterns. For Explore 9.2 at the start of this section, we calculated the data for MYP assessments where the score is between 1 and 8. However, if we were to grade a quiz with a percentage mark, the data could take values between 0 and 100. This would give far too many data values in the frequency table, so using groups would be more efficient.

When the data is continuous (for example: times or masses) it can only be recorded within a range, so these types of data are always grouped.

Thinking skills

Investigation 9.2

By using your GDC or other available software, represent each discrete data set with a bar chart.

Data set 1

Data	1	30	55
Frequency	1	20	6

Data set 2

Data	1	5	15	30	45	50	55
Frequency	1	6	10	20	13	9	6

Data set 3

Data	1	5	10	15	20	30	40	45	50	55
Frequency	1	6	7	10	11	20	14	13	9	6

Data set 4

Data	1	3	5	10	12	15	20	21	25	30	31	35	40	41	45	50	53	55
Frequency	1	4	6	7	8	10	11	12	15	20	19	17	14	14	13	9	7	6

- Describe what is happening to the bar chart and the gaps between the bars as the data values become closer together.
- What would the bar chart look like if we had data values starting at 1, increasing by 1 each time up to the maximum value of 55.
- What would the bar chart look like if we had data values starting at 0.5, increasing by 0.5 each time to the maximum value of 55?

When the data values become too numerous it makes sense to group them.

- How could Data set 4 be grouped? How would you communicate this?
- If the data values started at 0.5 and increased by 0.5 each time to the maximum value of 55, how could they be grouped? How would you communicate this?
- If the data values started at 0.05 and increased by 0.05 each time to the maximum value of 55, how could they be grouped? How would you communicate this?
- When does discrete grouped data need to be communicated in the same way as continuous data?

Hint

- When discrete data are grouped, they can be represented with different notation depending on the nature of the data.

  10–20 means that the data values in the group could range from 10 to 20 inclusive.

  21–30 means that the data values in the group could range from 21 to 30 inclusive.

  This notation does not work if we have discrete data values in between the groups, for example 10.5.
- Sometimes grouped discrete data is communicated with continuous data notation. It is important that we do not count a value in two different groups. This is overcome with the inequality notation:

  $10 < x \leq 20$ means that 10 is not included in the group, but 20 is included.

  $20 < x \leq 30$ means that 20 is not included in the group, but 30 is included.

  This notation ensures that 20.5 is accounted for in the second group, but 20 is not put into two different groups.

Connections

Data can be described by using the notation that is also used to communicate inequalities.

Worked example 9.3

The data organised in the frequency table below show the percentage scores of students from six different Grade 9 classes who completed a Physics quiz. As the data would have been rounded with small gaps between the data values, it can be represented as grouped data with inequality notation.

Three classes were told about the test ahead of time, and three were not.

Score	Frequency
$0 < x \leq 10$	2
$10 < x \leq 20$	16
$20 < x \leq 30$	25
$30 < x \leq 40$	7
$40 < x \leq 50$	7
$50 < x \leq 60$	8
$60 < x \leq 70$	7
$70 < x \leq 80$	16
$80 < x \leq 90$	27
$90 < x \leq 100$	5

a State the modal group (the group with the highest frequency).

b Estimate the mean.

c Determine the median group.

d Comment on the values.

Connections

Classes can be organised slightly differently. Differences in the results are minimal.

Solution

The data is grouped; therefore, we must identify the modal group as the group with the highest frequency. We must also use the midpoint of the group as a representative value of the group. This is why we have been asked to 'estimate' the mean. The data is collected in a frequency table, so we know how many values are in each group.

We can identify the modal group from the table.

We can find the midpoint of the data in each group by:

$$\text{midpoint} = \frac{\text{upper bound} + \text{lower bound}}{2}$$

Fact

The lowest value of a group is sometimes called the lower bound. The highest value of a group is sometimes called the upper bound.

We can find the mean of the data by using formula:

$$\text{mean} = \frac{\text{sum of (midpoint} \times \text{frequency)}}{\text{sum of the frequencies}}$$

The median is the data value in the middle of the observations when they are in numerical order. We need to determine the interval in which the middle value lies; this will be done by calculating the cumulative frequencies.

a The modal group is the group with the highest frequency, and this can be seen from the table: Modal group is $80 < x \leq 90$

b It is a good idea to organise calculations in a table when calculating the mean and determining the median group of grouped data:

Score (%)	Midpoint	Frequency	Midpoint × Frequency		Cumulative frequency
$0 < x \leq 10$	5	2	5×2	10	2
$10 < x \leq 20$	15	16	15×16	240	18
$20 < x \leq 30$	25	25	25×25	625	43
$30 < x \leq 40$	35	7	35×7	245	50
$40 < x \leq 50$	45	7	45×7	315	57
$50 < x \leq 60$	55	8	55×8	440	65
$60 < x \leq 70$	65	7	65×7	455	72
$70 < x \leq 80$	75	16	75×16	1200	88
$80 < x \leq 90$	85	27	85×27	2295	115
$90 < x \leq 100$	95	5	95×5	475	120
Totals		120		6300	

$$\text{mean} = \frac{\text{sum of (data value} \times \text{frequency)}}{\text{sum of the frequencies}}$$

$$= \frac{6300}{120} = 52.5$$

c The median is the data point that is halfway through the data. As there are 120 data values, the median will be between the 60th and the 61st data values. As for the non-grouped data we can find the position of the median value efficiently:

$$\text{position of the median value} = \frac{\text{total frequency} + 1}{2}$$

$$= \frac{120 + 1}{2}$$

$$= 60.5$$

Therefore, the median value is the data value in position 60.5

For grouped data we can identify the group which contains the median data value and refer to the median group rather than the median value. For this example, the 60.5th data value lies in the data group $50 < x \leq 60$. Our median group is therefore $50 < x \leq 60$.

d In this example neither the mean value nor the median value is within the modal group.

The modal group is $80 < x \leq 90$, the estimate of the mean is 52.5 and the median group is the interval $50 < x \leq 60$. As the modal group is higher than the median group and the mean is also outside of the modal group, it suggests that the data is not symmetrical around the mean. This implies that there could be a difference in the test results depending on the class. This observation justifies further investigation for underlying patterns to determine if the test was fair.

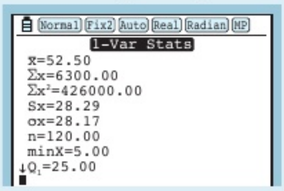
Looking back, the mean and median group are similar, but different from the modal group. Is there a valid reason for this? The data in the table has higher frequencies at the upper end and the lower end than in the middle. This suggests that the mean and median could be in between the two most frequent groups.

From the data, we could infer that the students who were told about the quiz did better than the students who weren't. But this isn't clear, as we do not know which students are in each group. It would be fair to the students who didn't know about the test to investigate this further.

Hint

You can use your GDC to perform your calculations by using the midpoints as the data values and using the one variable statistics option. The GDC will give you a more accurate estimate of the median value.

A sample output



Reflect

Thinking skills

If you estimate the mean by using the midpoint to represent the data values in each group, does this affect any conclusions you make about the data?

Can the data be represented in another way to look for patterns and explore the fairness of the test?

Can you think of another method for estimating the median of grouped data?

Can you find the range for grouped data?

Practice questions 9.2.3

1 For the data sets below:

i state the modal group

ii determine the median group

iii estimate the mean.

a

x	0–10	11–20	21–30	31–40	41–50
Frequency	5	7	2	4	6

b

x	$5 < x \le 10$	$10 < x \le 20$	$20 < x \le 30$	$30 < x \le 40$
Frequency	2	9	8	3

2 A local supermarket delivers online shopping for a charge that is determined by the distance of the delivery address from the store. The table below shows the charge per kilometre and the number of deliveries the store makes per day.

a Estimate the mean, determine the median group, and state the mode for the cost of deliveries.

b Estimate the mean, determine the median group, and state the mode for the distance travelled by the delivery drivers.

c Calculate the range for the charge.

d Explain why you cannot find the range for the distance travelled.

Delivery distance (km)	Charge (€)	Number of deliveries
$0 < x \le 5$	5	32
$5 < x \le 10$	7.5	12
$10 < x \le 15$	10	6
$15 < x \le 20$	15	2

3 The data in the table below is incomplete. Given that the mean is 7.5, find the missing frequency value a.

x	2	5	10	12
Frequency	4	a	4	6

4 The data in the table below is incomplete. Given that the modal group is $10 < x \le 20$, what could the missing frequency value be?

x	$5 < x \le 10$	$10 < x \le 20$	$20 < x \le 30$	$30 < x \le 40$	$40 < x \le 50$
Frequency	2	9	?	7	6

5 The data in the table below is incomplete. Given that the mean is 54.8, what is the missing frequency value a?

x	$0 < x \le 20$	$20 < x \le 40$	$40 < x \le 60$	$60 < x \le 80$	$80 < x \le 100$
Frequency	2	5	6	9	a

Challenge Q6

6 The data in the table below is incomplete. An estimate of the median is 25, an estimate of the mean is 24.375, there are 2 modal groups and $a \neq b$. What are the missing frequency values a and b?

x	$0 < x \le 10$	$10 < x \le 20$	$20 < x \le 30$	$30 < x \le 40$	$40 < x \le 50$
Frequency	1	a	5	4	b

9.3 Representing data

The data explored so far in this section have been presented in list form or organised into frequency tables. Data can also be presented in charts or graphs. The form in which you choose to display the data depends on the type of data and on what you are trying to show.

9.3.1 Representing discrete data

Explore 9.3

UNICEF published the following data regarding the percentage of children of *lower secondary school age* around the world who are not receiving education (the out-of-school rate).

Region	Females	Males
East Asia and Pacific	8	9
Europe and Central Asia	4	3
Eastern Europe and Central Asia	5	4
Western Europe	2	2
Latin America and Caribbean	7	7
Middle East and North Africa	15	12
North America	1	1
South Asia	17	18
Sub-Saharan Africa	37	34
Eastern and Southern Africa	37	32
West and Central Africa	40	37

One way of displaying this data is in the form of a **back-to-back stem-and-leaf plot.**

Can you compare the male and female data using a stem-and-leaf plot?

The data in the table below lists the percentage out-of-school rates for students of *upper secondary school age*.

Region	Females	Males
East Asia and Pacific	15	23
Europe and Central Asia	10	10
Eastern Europe and Central Asia	14	14
Western Europe	6	7
Latin America and Caribbean	21	24
Middle East and North Africa	36	31
North America	5	5
South Asia	49	47
Sub-Saharan Africa	61	54
Eastern and Southern Africa	60	54
West and Central Africa	61	53

By constructing a suitable plot for the upper school data, determine whether there is a difference between the lower and upper school data.

Hint

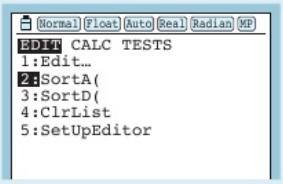
A stem-and-leaf plot represents discrete data in a form that helps us to identify the mode and the median.

Plotting two sets of data on a back-to-back stem-and-leaf plot allows us to visually identify any similarities and differences between the two data sets.

Hint

Entering the data into your GDC and using the Sort function will put your data in numerical order.

A sample output



Communication skills

Thinking skills

Worked example 9.4

The WHO/UNICEF Joint Monitoring Program for Water Supply, Sanitation and Hygiene (June 2019) published the following data, which gives the percentage of the population in different world regions who have access to basic sanitation services in urban and rural areas.

Region	Urban	Rural
East Asia and Pacific	91	75
Europe and Central Asia	98	93
Eastern Europe and Central Asia	97	88
Western Europe	99	99
Latin America and Caribbean	91	69
Middle East and North Africa	95	82
North America	100	100
South Asia	70	53
Sub-Saharan Africa	45	22
Eastern and Southern Africa	49	22
West and Central Africa	41	21

a Complete a back-to-back stem-and-leaf plot for the urban and rural data.

b Use your stem-and-leaf plot to identify the mode and the median for each data set.

c Calculate the mean value and the range for the urban data and the rural data.

d By considering your statistic values from parts b and c, discuss whether there are differences between the level of access for people in urban areas and rural areas around the world.

Solution

The raw data is given in a table, so the stem needs to be identified and the back-to-back stem-and-leaf plot constructed. Constructing a back-to-back stem-and-leaf plot enables the two sets of data to be compared. The maximum and minimum values, the measures of central tendency and the ranges can be determined and compared.

The data range from 21 to 100 will determine the stems for the diagram. The mode and median can be identified from the data once it is organised into the stem-and-leaf plot, because this puts the data in order.

The mean and range can be calculated using the formulae:

$$\text{mean} = \frac{\text{sum of all the data values}}{\text{the number of data values}}$$

$$\text{range} = \text{maximum value} - \text{minimum value}$$

Compare the statistic values and identify similarities and differences. Suggest explanations for the observations.

a Completing the stem-and-leaf plot:

In this example, the stem is the tens part of the data value.

Urban		Rural
	2	1 2 2
	3	
9 5 1	4	
	5	3
	6	9
0	7	5
	8	2 8
9 8 7 5 1 1	9	3 9
0	10	0

Urban: **1 | 4 = 41**

Rural: **5 | 3 = 53**

where 5 is the stem and 3 is the leaf

The unit part of the number is the leaf; for example this digit is the 1 in the data value 21 for the Rural data.

In the back-to-back plot, the leaf part for each data set is placed on either side of the stem.

The 1 to the left of the stem, is the unit part of the data value 41 in the Urban data.

The leaves must be written in increasing numerical order moving from the stem outwards, to the left or to the right.

Fact

The key represents the stem and the leaf for the data set you are using. The leaf is always the digit on the far right of the number. The stem is always the digit or digits to the left of the leaf. For example, 12.5 would have a leaf of 5 and a stem of 12.

b The mode for the Urban data can be identified as 91 and the Rural mode is 22 as there are two of these data values.

It is important to remember what the leaves 1 and 2 represent. You must write your answer as the data value, not just the leaf. Always include the stem part in your answer.

The median can be identified as the middle value. As the data is already sorted into numerical order, the position of the median data value can be found efficiently using:

$$\text{position of the median value} = \frac{\text{total frequency} + 1}{2}$$

For example, the Urban data:

$$\text{position of the median value} = \frac{11 + 1}{2}$$

$$= 6$$

For the Urban data the median is 91, for the Rural data the median is 75.

c The mean and range are calculated by using the formulae:

Urban:

$$\begin{aligned}\text{mean} &= \frac{41+45+49+70+91+91+95+97+98+99+100}{11}\\ &= \frac{876}{11}\\ &= 79.64\\ \text{range} &= 100-41\\ &= 59\end{aligned}$$

Rural:

$$\begin{aligned}\text{mean} &= \frac{21+22+22+53+69+75+82+88+93+99+100}{11}\\ &= \frac{724}{11}\\ &= 65.82\\ \text{range} &= 100-21\\ &= 79\end{aligned}$$

d If we summarise the values in a table, we can compare the two sets of data more easily.

	Mean	Median	Mode	Range
Urban	79.64	91	91	59
Rural	65.82	75	22	79

The statistics values for the rural data are all lower than for the urban data, suggesting that there is less access to basic sanitation for people living in rural areas than for those in urban areas.

The measures of central tendency are not equal for either set of data. This suggests that both sets of data are not symmetrical, implying differences between access to sanitation in different regions of the world in both urban and rural areas. This is confirmed by looking at the stem-and-leaf plot. For both data sets there are data values at the upper end and the lower end of the range, with fewer in the middle. The range of the rural data is larger than for the urban data, which suggests that there is a greater difference in access to sanitation in different regions of the world in rural areas compared with urban areas.

Looking back, the conclusions drawn by comparing the statistics values support the data in the table as the rural values appear to be lower than the urban. Representing the data in the back-to-back stem-and-leaf plot enabled the differences between the data sets to be identified. As the data was organised into numerical order, it allowed the mode and median to be identified more easily. The original table also shows a difference in the values between the different world regions. This supports the observations from the analysis.

Reflect

Would displaying the data in a bar chart provide the same information as the stem-and-leaf plot?

Practice questions 9.3.1

1 Represent the rainfall and sunshine data in separate stem-and-leaf plots.

Rainfall per week (mm)	1	18	4	2	0	2	23	3
Sunshine per week (hours)	15	0	19	20	70	21	0	20

Explain why these data sets are displayed as separate stem-and-leaf plots and not back-to-back.

2 As part of research for his Environmental Science degree project, Tomás measured the level of organic residue material (mg per L) in the water flowing from 10 paper mills before and after treatment.

His results are shown in this table.

Before	41	44	49	54	54	55	56	63	65	73
After	18	19	22	23	34	38	38	41	44	51

a Represent the data in a back-to-back stem-and-leaf plot.

b For each set of data:

i identify the mode

ii find the median

iii calculate the mean

iv calculate the range for the data before and after treatment.

c What can Tomás conclude from these test results?

3 Bill thinks that the Grade 10 Physics test was more difficult than the Chemistry one. He collects test data for the students who took both tests.

Physics (%)	61	48	58	35	45	72	36	56	48	51	60	79	54	27	44
Chemistry (%)	73	89	76	58	39	91	67	59	71	44	68	73	79	61	73

a Represent the data in a back-to-back stem-and-leaf plot.

b For each set of data:

i identify the mode

ii find the median

iii calculate the mean

iv calculate the range.

c What can Bill conclude from these test results?

4 Use the data represented in the stem-and-leaf plot below.

Set A		Set B
3 2	10	
8 7 2	11	
6 5 3 3 3	12	
8 8 5 4	13	4
8 4 1	14	8 9
1 0	15	1 7
2	16	1 6 9 9 9 9
	17	5 5 6 7 8
	18	1 2 2 3 4

Set A **1 | 5 = 151** **Set B** **13 | 4 = 134**

a For each data set A and B:

i identify the mode

ii find the median

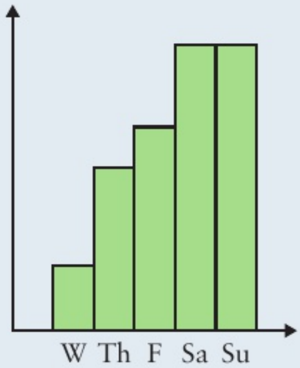
iii calculate the mean

iv calculate the range.

b What conclusions could you make about the data?

c What could these data sets represent?

5



Simone has been given a chart representing visitors to a gallery during the opening times. Her supervisor has asked her to represent the data in a stem-and-leaf plot, but there is a problem. The chart doesn't have clearly labelled scales on the axes. She knows that the mean of the data is 14.4, the median is 15, the range is 15 and the mode is 20. There are 5 data values as the gallery opens for 5 days per week and the total number of visitors was 72.

a Construct a frequency table of the data and represent the data in a stem-and-leaf plot.

b Does the stem-and-leaf plot identify anything that the bar chart doesn't? Was there a benefit to representing the data in this way?

c If Simone didn't know the value of the range, explain how this would have affected the solution.

Challenge Q5

Thinking skills

9.3.2 Representing grouped data

Explore 9.4

In Section 9.2.3, we explored the percentage scores for a Physics test for six Grade 9 classes. The descriptive statistics identified an estimate of the mean as 52.5, the median group as $50 < x \leq 60$ and the modal group as $80 < x \leq 90$. Analysis of the values identified the data as asymmetrical because the values were different. This also implied that the test might not have been fair. The data is given in the following table.

Score	Frequency
$0 < x \leq 10$	2
$10 < x \leq 20$	16
$20 < x \leq 30$	25
$30 < x \leq 40$	7
$40 < x \leq 50$	7
$50 < x \leq 60$	8
$60 < x \leq 70$	7
$70 < x \leq 80$	16
$80 < x \leq 90$	27
$90 < x \leq 100$	5

Can you represent the data by a suitable chart? Explain your choice of chart and whether or not it supports the statistics values.

Fact

A bar chart is usually used to represent categorical data and is drawn with separate, distinct bars that correspond to one value on the horizontal axis.

Fact

For continuous data when drawing a bar chart, the bars touch, so representing the continuous nature of the data. The bars correspond to a range of values on the horizontal axis called **classes**, and the class width is the difference between the upper and lower boundary.

Thinking skills

- Data can be represented in different types of chart to help identify patterns and trends. The choice of chart depends on the type of data.
- Stem-and-leaf plots are suitable for small discrete data sets, or continuous data sets that are rounded with a relatively small range. If the data set is large and organised with large frequency values or the range is too large, the stem-and-leaf plot becomes too crowded, or has too many stems and the patterns are hidden.
- Representing grouped data in the form of a histogram would be more effective for identifying patterns for these data sets.

Fact

When a bar chart represents grouped discrete data, it is sometimes drawn with gaps between the bars. If the gaps between the bars are small, it will be drawn with the bars touching and will be called a histogram.

Connections

Concept of an inequality and communication using the inequality signs.

Worked example 9.5

The OECD publishes figures for world average salaries each year. The data for 2019 (given in US$) are summarised below. The table shows the average salary range and the number of countries with this average salary range.

Salary US$	Number of countries
$10{,}000 \le S < 20{,}000$	1
$20{,}000 \le S < 30{,}000$	8
$30{,}000 \le S < 40{,}000$	6
$40{,}000 \le S < 50{,}000$	7
$50{,}000 \le S < 60{,}000$	9
$60{,}000 \le S < 70{,}000$	4

a Explain why these data cannot be represented by a stem-and-leaf plot.

b Explain why a histogram with the bars touching would be most suitable for this type of data.

c Represent the data with a histogram and describe any features.

Solution

The data given in the table is continuous. A stem-and-leaf plot is more suitable for discrete data. The data is grouped because it is continuous and there is no distinct separation of the groups, so a histogram is used.

Explain why continuous data cannot be easily represented by a stem-and-leaf plot and why it can be represented by a histogram.

Hint c

You can use online graphical calculators to make statistics charts. Explore Desmos:

and GeoGebra:

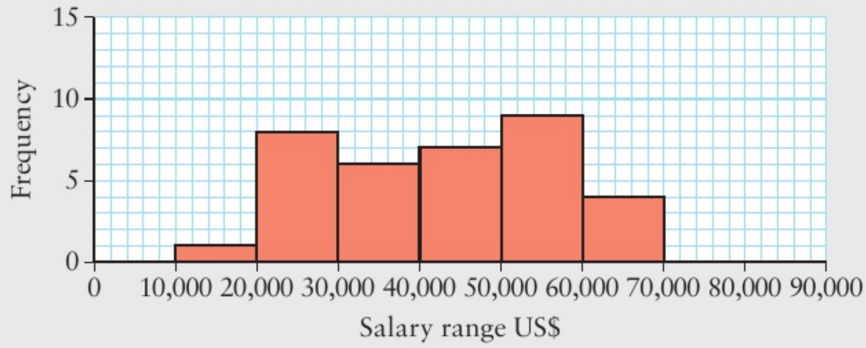
graphing calculators to represent this data.

Construct a histogram with group widths of 10,000 on the horizontal axis and a frequency scale of 0 to 10 on the vertical axis.

a The data is grouped and therefore it cannot be shown in a stem-and-leaf plot. Although the midpoints could be used to represent each group, this would only lead to repeats of the same value and this makes the chart meaningless.

b The histogram is a suitable choice for representing the data because it is organised into groups and the data is continuous. For example, there is no distinction between a salary of $49,999.99 and $50,000, and this can be represented by bars that touch.

c Constructing the histogram:



The histogram is a visual representation of the data and it confirms that the lowest average salary group is approximately $50,000 less than the highest average salary.

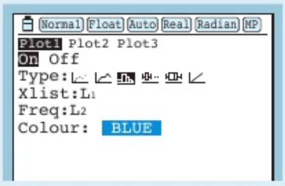
It also confirms that the lowest average salary group has the lowest frequency.

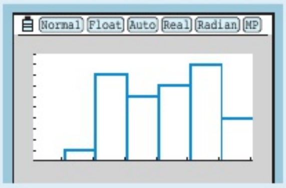
Looking back, the histogram consists of bars that touch to represent the continuous nature of the data. The diagram shows that the modal group is the $(50,000 \leq S < 60,000)$ range.

Hint

You can use your GDC to represent your data as a histogram to check your diagram.

A sample output





Reflect

If the midpoints of the bars on a bar chart are joined by straight lines, the graph is called a **frequency polygon**. When might it be useful to use a frequency polygon as a representation of grouped data?

Practice questions 9.3.2

1 Represent each set of continuous data given in the tables with a histogram.

a

T	Frequency
$0 \le T < 5$	1
$5 \le T < 10$	6
$10 \le T < 15$	11
$15 \le T < 20$	12
$20 \le T < 25$	5
$25 \le T < 30$	1

b

W	Frequency
$0 \le W < 100$	10
$100 \le W < 200$	3
$200 \le W < 300$	11
$300 \le W < 400$	2
$400 \le W < 500$	14
$500 \le W < 600$	1

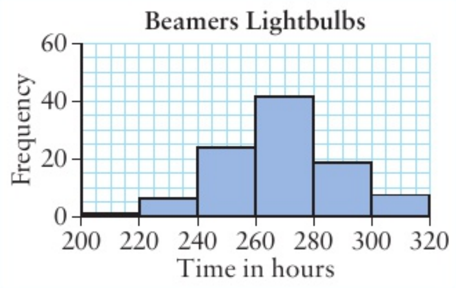
2 As part of his Sports Science course, Marc measured the time it took for the members of the football squad to complete a lap of the practice field. He recorded the time in seconds in the table below.

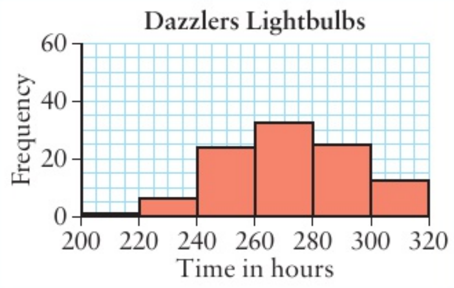
T	Frequency
$0 \le T < 30$	0
$30 \le T < 60$	1
$60 \le T < 90$	10
$90 \le T < 120$	3
$120 \le T < 150$	1

Represent the data with a histogram.

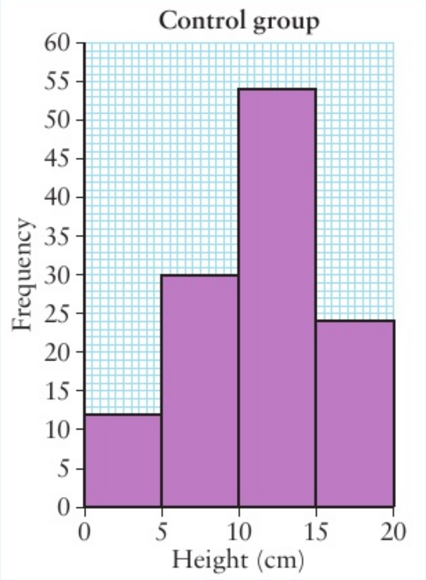
3 The histograms below represent the data published by two lightbulb manufacturers, Beamers and Dazzlers. The charts represent the lifetime of 100 light bulbs. Which company is producing the most reliable lightbulbs?

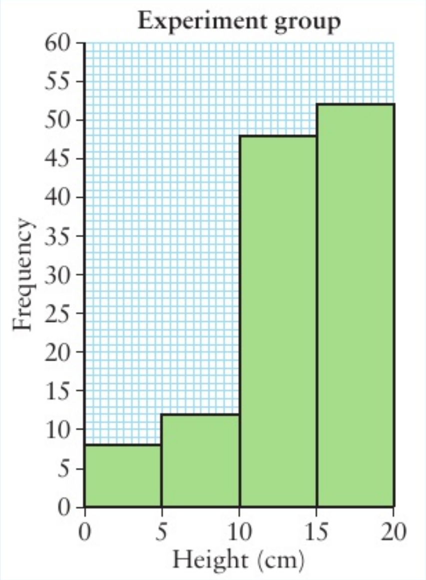
Use the information from the bar charts to justify your answer.





4 Raman's Biology project explored the effect of different nutrients on plant growth. He measured the height of his control group and experiment group, recorded the data on a sheet of paper and then represented the results with two histograms, as shown below.





Unfortunately, Raman lost the sheet of paper before he was able to calculate the statistics values.

a Determine the modal group for the control and experiment data.

b By using the information in the histograms, represent Raman's control and experiment data in separate frequency tables.

c Determine the median group for the control and experiment data.

d Estimate the mean for the control and experiment data.

5 The continuous data investigated so far has always been organised into groups with equal widths. But what happens if the data is not collected in groups of equal widths?

a Construct a histogram with unequal widths using the data in the table below.

T	Frequency
$0 \leq T < 25$	100
$25 \leq T < 50$	250
$50 \leq T < 70$	200
$70 \leq T < 80$	100

b Critically evaluate the chart that is produced. Is it a fair representation of the data?

Research skills

c Research **frequency density** from a reputable online source and construct a second histogram with the frequency density value on the vertical axis.

d Compare the two histograms and explain why histograms with unequal group widths are plotted with frequency density rather than frequency on the vertical axis.

Self assessment

- I can classify data as discrete or continuous.
- I can identify the mode of a set of discrete data.
- I can find the median of a set of discrete data.
- I can calculate the mean of a set of discrete data.
- I can identify a leading question in a survey.
- I can design questions that are concise and targeted to the survey needs.
- I can explain the importance of eliminating bias from a survey.
- I can identify data as primary or secondary.
- I can critically evaluate the source of secondary data.
- I can organise data into frequency tables.
- I can identify the modal group of a grouped data set.
- I can calculate the mean of data from a frequency table.
- I can determine the median of data from a frequency table using the cumulative frequencies.
- I know that the mean, median and mode are measures of central tendency.
- I can find the range of a data set.
- I know that the range is a measure of dispersion.
- I can estimate the mean of a set of grouped data from a frequency table.
- I can estimate the median of a set of grouped data from a frequency table by using cumulative frequencies.
- I can represent discrete data in the form of a stem-and-leaf plot.
- I can represent two sets of data on a back-to-back stem-and-leaf plot and compare them.
- I can determine the mean, median and mode of a data set from the stem-and-leaf plot.
- I can explain why a stem-and-leaf plot is not suitable for displaying grouped data.
- I can represent grouped data with a histogram.
- I can explain why a histogram is drawn with bars touching to represent continuous data.
- I can use a histogram to identify the modal group of a grouped data set.
- I can identify key information from a histogram to be able to represent the data in a frequency table.

Check your knowledge questions

1 State whether each set of data is discrete or continuous.

a The circumference of your head

b Your hat size

c The number of cars bringing students to school in the morning

d The time taken to travel to school

2 Find the mean, median, mode and range of the following data sets.

a

13	15	18	23	31	35	48	48	60

b

9	5	12	21	18	35	28	12	60

3 State which question, A or B, is the better question for each survey. Explain your choice.

Survey information required	Question A	Question B
a The age of the person completing the survey	What age are you?	Which age group matches your age? ◯ under 18 ◯ 18 to 60 ◯ over 60
b Do people donate to charities?	I give part of my monthly allowance to charity. Do you?	Do you donate money to a charity? yes ◯ no ◯
c If a hotel was ok	Was everything as you expected? yes ◯ no ◯	Did you go into town for the evening? yes ◯ no ◯
d If they have seen a film	Have you seen Star Wars? yes ◯ no ◯	Yoda like do you? yes ◯ no ◯

4 Which of the following would require you to generate primary data and which would require secondary data? Explain your choice and how you would source the information.

a You want to know the mean height of your classmates.

b You want to know the mean age of the people who live in your town.

c Your Individuals and Societies teacher sets you a group project on the types of land use in Rio de Janeiro, Brazil.

d Your Biology teacher sets you the task of classifying the wildlife in the school pond.

5 The data below are the percentage scores in a Grade 9 Mathematics quiz.

89	45	67	89	21	91	67	98	34	54	67	28	74	57	69
66	56	46	76	34	92	38	46	36	76	55	64	68	91	39
49	51	58	23	89	82	75	69	35	64	72	87	35	44	23
44	34	54	23	98	66	89	90	23	36	52	33	42	32	67
34	54	66	79	35	80	49	68	79	45	74	81	48	65	47

a Organise the data into a grouped frequency table with groups of width 10 percent.

b Identify the modal group and the median value.

c Calculate the mean score.

d Calculate the range of the data

Hint

Find the median group and estimate the median using the midpoint of the group.

6 Copy the data table below and complete the additional column to record the cumulative frequencies.

Use your cumulative frequencies to find the median group of the data.

W	Frequency	Cumulative frequency
$0 \le W < 100$	5	
$100 \le W < 200$	13	
$200 \le W < 300$	11	
$300 \le W < 400$	14	
$400 \le W < 500$	25	
$500 \le W < 600$	10	
$600 \le W < 700$	9	
$700 \le W < 800$	5	

7 A Grade 9 teacher gave a mathematics test to two classes, A and B. The results of the test are given in the table.

Class A	89	45	67	89	21	91	67	98
Class B	66	56	46	76	34	92	38	46

Class A	34	54	67	28	74	57	69	
Class B	36	76	55	64	68	91	39	

a Represent the data in a back-to-back stem-and-leaf plot.

b Use the stem-and-leaf plot to identify the range, the median and the mode for each of the two classes.

c Calculate the mean for each of the two classes.

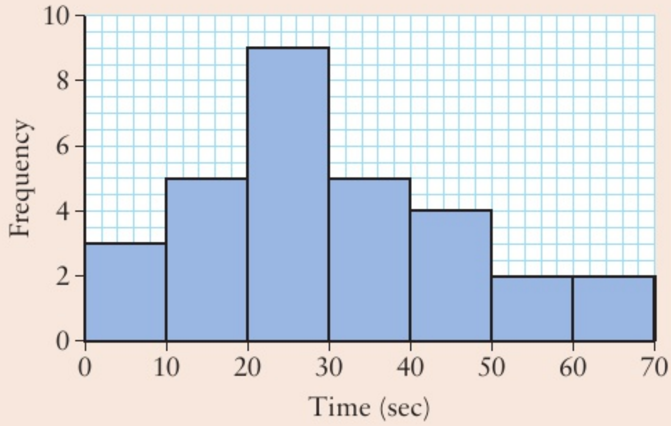
d Compare the results of the mathematics test for the two classes.

8 Represent the data below on a histogram.

W	Frequency
$0 \leq W < 100$	5
$100 \leq W < 200$	13
$200 \leq W < 300$	11
$300 \leq W < 400$	14
$400 \leq W < 500$	25
$500 \leq W < 600$	10
$600 \leq W < 700$	9
$700 \leq W < 800$	5

9 Draw a histogram to represent the data in your grouped data frequency table for question 5. Does the chart correspond to the statistics values you calculated?

10 A data set is represented in the form of a histogram.



a Use the histogram to generate a frequency table.

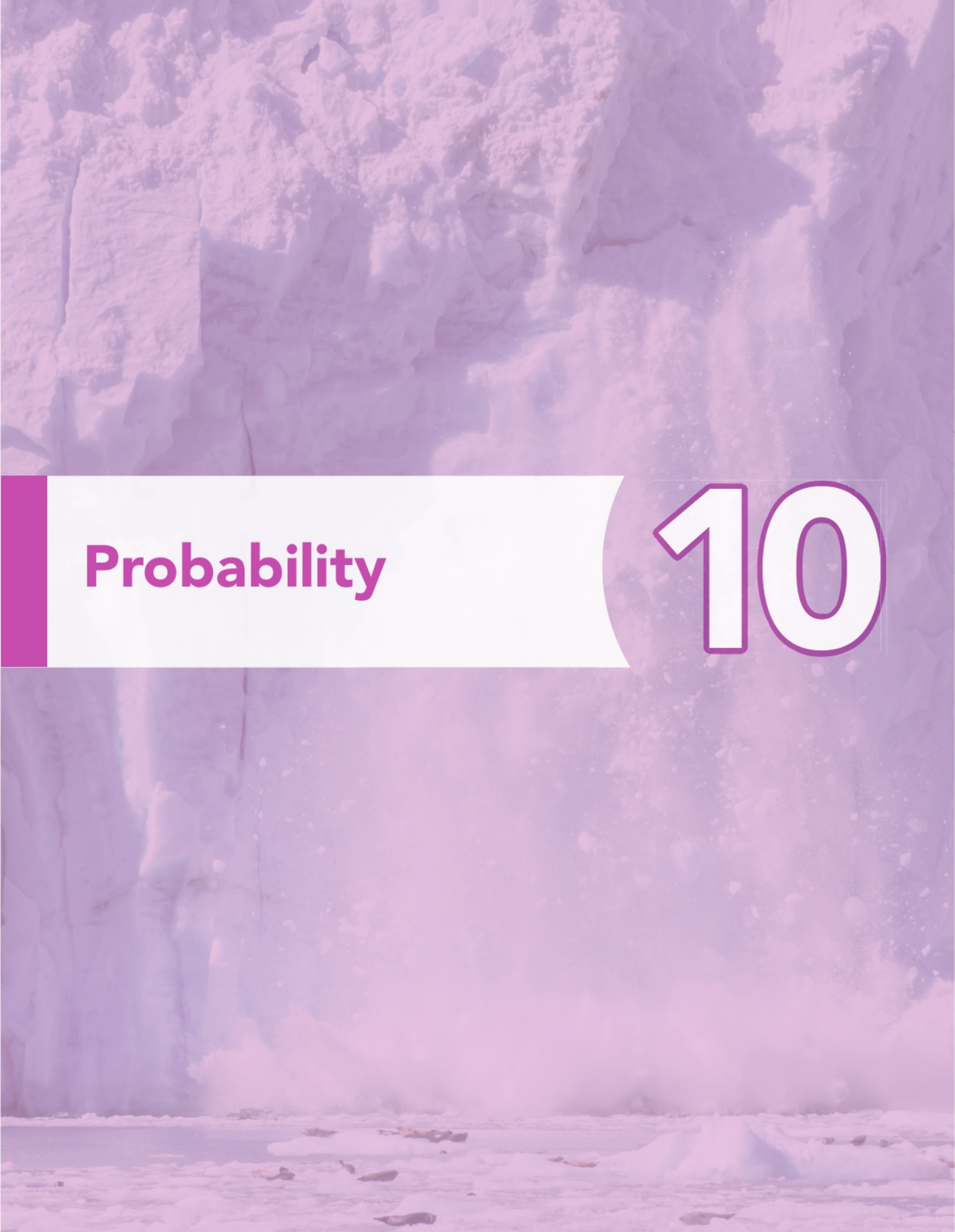
b Use your frequency table to:

i determine the modal group

ii determine the median group

iii calculate an estimate of the mean.

c Is it necessary to move from the histogram representation of the data to the frequency table to find the statistics values? Explain your reasoning.



10 Probability

10 Probability

KEY CONCEPT

Logic

RELATED CONCEPTS

Models, Pattern, Representation, Validity

GLOBAL CONTEXT

Globalisation and sustainability

Statement of inquiry

By observing patterns to model chance we can make logical decisions about our global environment.

Factual

- How can we calculate the probability of repeated simple events?

Conceptual

- How do theoretical and experimental probability differ?

Debatable

- Are the models that probability offers always justified?

Do you recall?

1 Perform these calculations:

a $\frac{1}{6}+\frac{1}{3}$ b $\frac{3}{4}-\frac{1}{2}$ c $\frac{1}{4}+\frac{2}{7}$ d $\frac{4}{5}-\frac{1}{2}$

2 Perform these calculations:

a $0.92 + 0.03$ b $0.56 - 0.2$ c $0.14 + 0.5$ d $0.63 - 0.49$

3 Find the probability of these simple events.

a What is the chance of rolling a 6 on a six-sided dice?

b What is the chance of a flipping a coin and it landing on heads?

4 Find the probability of these outcomes.

a Rolling a prime number on a six-sided dice.

b Drawing a red card from a standard pack of playing cards.

10.1 Theoretical probability

10.1.1 Reviewing the basics

'The theory of probability is the only mathematical tool available to help map the unknown and the uncontrollable. It is fortunate that this tool, while tricky, is extraordinarily powerful and convenient.' – Benoit Mandelbrot

Explore 10.1

Can you give an example of an event that:

a is certain

b is likely to happen

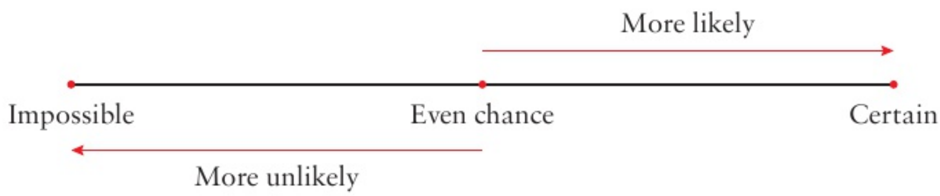
c has an even chance of happening

d is unlikely to happen

e is impossible?

Looking at the events you listed, can you calculate or estimate a numerical value to fit with the event?

Probability helps us predict the likelihood of outcomes of selected experiments. The chance of a certain **outcome** happening can range from impossible to certain.



You might remember that the outcomes that are easiest to predict are those of **simple events**. These are events where there is only one way to get a specific outcome. For example, when flipping a coin, the only two outcomes are heads and tails. On an unbiased coin, this would result in even chances of either outcome.

Worked example 10.1

A biased coin is flipped. Calculate the probability of landing on heads if:

a the coin lands on tails in three out of four flips

b heads is twice as likely to occur as tails

c both sides of the coin are tails.

Solution

The coin is biased meaning the odds are different from the standard 50/50 odds.

a Let's convert the given information to find the likelihood of tails. After that we can find the probability of heads.

$P(\text{Tails}) = \frac{3}{4}$ — Use probability notation where possible.

$P(\text{Heads}) = 1 - P(\text{Tails})$ — Since the only options are heads or tails, the remaining probability must be heads.

$P(\text{Heads}) = 1 - \frac{3}{4} = \frac{1}{4}$ — Substitute the known value for P(Tails).

Looking back, if the coin lands on tails three times out of four, then there is only one time left for heads. Therefore, a probability of one fourth must be true.

b First we will use the given information to find a ratio between heads and tails. After that we can find the probability of heads, by looking at the total parts.

$2:1$ — Ratio of heads to tails, giving a total of 3 parts.

$P(\text{Heads}) = \frac{2}{3}$ — This means the ratio of heads to total parts is $2:3$

Looking back, the question says heads is twice as likely, which make sense since $\frac{2}{3}$ is twice as much as P(Tails), which must be $\frac{1}{3}$

c $P(\text{Heads}) = 0$

Looking back, since there is no side with heads, then this is an impossible event and its probability must be zero.

The outcomes of heads or tails is an excellent example of complementary events. Complementary events are opposite events. Together, the sum of their probabilities is equal to 1 or 100%. This means that two complementary events together make up all the possible outcomes. We used this fact to our advantage in Worked Example 10.1.

Complementary events can also occur in events that are not simple. For example, the probability of rolling an odd number on a dice versus rolling an even number on a dice. Or the probability of the weather being sunny versus the probability of it not being sunny. We can also write the probability of complements in the symbolic form: $P(X) + P(X') = 1$, where $P(X')$ represents the probability of X **not** happening.

When we have to predict more complicated outcomes, it can be useful to list out the **sample space**. This is a list of all possible outcomes of a probability experiment. This allows us to calculate probabilities by using the following formula: $\mathrm{P}(\textit{outcome } X) = \dfrac{\text{number of ways outcome } X \text{ can occur}}{\text{number of all possible outcomes}}$

We can shorten this using the notation: $\mathrm{P}(X) = \dfrac{n(X)}{n(S)}$ where $n(\ )$ stands for 'number of' and S stands for 'sample space'.

Worked example 10.2

You roll an eight-sided dice.

a List the sample space.

b How many outcomes are in the sample space?

c Use this to help you calculate the probability of rolling:

- **i** an even number
- **ii** a number greater than 6
- **iii** a prime number
- **iv** a factor of 12.

Solution

a $\{1, 2, 3, 4, 5, 6, 7, 8\}$ — The sample space of rolling an 8-sided dice.

b 8 — There are 8 sides for the dice to land on.

c **i** $\{1, 2, 3, 4, 5, 6, 7, 8\}$ — 4 numbers in the sample space are even.

$\mathrm{P}(X \text{ is even}) = \dfrac{4}{8}$ — The total number in the sample space is 8.

$= \dfrac{1}{2}$ — Simplify if possible.

ii $\{1, 2, 3, 4, 5, 6, 7, 8\}$ — Only 2 numbers are greater than 6.

$\mathrm{P}(X > 6) = \dfrac{2}{8}$

$= \dfrac{1}{4}$

iii $\{1, 2, 3, 4, 5, 6, 7, 8\}$ — There are 4 prime numbers.

$\mathrm{P}(X \text{ is prime}) = \dfrac{1}{2}$ — We know this is half the sample space.

iv $\{1, 2, 3, 4, 5, 6, 7, 8\}$ — There are 5 factors of 12.

$\mathrm{P}(X \text{ is a factor of } 12) = \dfrac{5}{8}$

Reminder

The **sample space** is the list of all possible outcomes of a probability experiment.

Practice questions 10.1.1

1 Classify the following events as impossible, very unlikely, unlikely, equally likely, likely, very likely or certain:

a a snowstorm in Brazil in the summer

b lightning during a thunderstorm

c a wildfire in the summer in California

d sun on a cloudless day.

2 You choose a card at random from the set below. Using your probability vocabulary, classify the event as impossible, very unlikely, unlikely, equally likely, likely, very likely or certain.

a The card you draw is a heart or diamond.

b The card is a jack or queen.

c The card is a numbered card.

d The card is a club.

e The card is an ace.

3 You choose a card at random from the set above. Use your probability knowledge to calculate the probability of drawing:

a an even number

b a heart

c a factor of 30

d a queen

e a card that is not an ace.

4 You roll a six-sided dice.

a List the sample space.

b How many outcomes are in the sample space?

c Use this to help calculate the probability of rolling:

i an odd number

ii a number less than 5

iii a prime number

iv a factor of 10.

5 A box has 3 clear marbles, 2 blue marbles and 4 green marbles.

a Draw a picture to visualise this situation.

b How many options are in the sample space for this probability experiment?

c Calculate the probability of randomly drawing:

i a clear marble

ii a blue marble

iii a green marble.

6 A jar with 2 milk, 5 dark and 7 white chocolates is in the pantry. John takes one at random.

a How many outcomes are in the sample space of this probability experiment?

b Which chocolate is most likely to be chosen? Calculate the probability of choosing it.

c Which chocolate is least likely to be chosen? Calculate the probability of choosing it.

7 A group of students is drawing a name at random out of a hat to win a prize. Five students are in the running for the prize: Ariadne, Benji, Caroline, Daniel and Elissa.

a Write out the sample space of this probability experiment.

b What is the probability that Elissa wins the prize?

c What is the probability that the student's name begins with a vowel?

d What is the probability that the student's name ends with a vowel?

8 A biased coin is flipped. Calculate the probability of it landing on tails if:

a the coin has been observed to land on tails in an average of 2 out of 3 flips

b tails is three times as likely to occur as heads

c both sides of the coin are tails.

Challenge Q9

9 You roll a six-sided dice. However, unlike a standard six-sided dice, this dice has no number 1, and instead shows the number 6 on two faces instead of just one.

a Write the sample space of rolling this dice.

b Find the probability of rolling:

i the number 2

ii an odd number

iii a prime number

iv a factor of 6.

10 Decide whether these probability statements are true, false or uncertain, giving a reason for your answer.

a There are 12 months in a year, so the probability of being born in December is 1 in 12.

b Because I passed my last Spanish test, it is certain I will pass the next one.

c There are 26 letters in the English alphabet, so there is a 1 in 26 chance of having a name starting with Z.

d If you roll three even numbers in a row, the next dice roll will very likely be odd.

10.1.2 Repeated probability events

Investigation 10.1

Imagine rolling two dice simultaneously. What would the sample space look like? How might that help us calculate our new probabilities?

1 Draw out a 6-by-6 table on grid paper, like the one below, labelling the rows and column with 1 through 6.

First dice

Second dice	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

- a What does this grid represent?
- b Notice there are two places where 1 and 6 intersect. What is the difference between the two rolls? Why is the difference significant?

2 Using your grid from question 1:

- a mark with a ✕ all the possibilities for both dice to show the same number
- b mark with an O all possibilities for the sum of the dice to be 8
- c shade or highlight all the combinations whose product is 6
- d describe at least one pattern you observed.

3 Using your answers for question 2, calculate:

- a the total possible number of roll combinations
- b the probability of rolling:
  - i the same number on both dice
  - ii a sum of 8
  - iii a product of 6
  - iv a 6 on both dice.

4 Use your answers from the previous parts to answer the following:

- a How are questions 2 and 3 related? Justify your thinking using at least one example.
- b How is the number of possible outcomes of rolling two dice related to the sample space of a single roll on a six-sided dice? Justify your thinking.

Reflect

Look back at Investigation 10.1 and answer the following questions.

How did the visual tool of the grid help you complete the questions? Give at least one specific suggestion.

Is drawing out a sample space an effective tool to solve probability questions? Why or why not?

Whether you are trying to predict daily weather patterns or playing a game of Yahtzee, real-world probability experiments tend to repeat themselves. Repeating a probability experiment can obviously lead to a greater combination of outcomes. However, the mathematics behind it is easier than you think, at least when the original probability stays the same.

As you saw in Explore 10.1, one way to account for the added repetitions is to draw or write out their sample space. In Explore 10.1 this was done with a grid, but this could also be done with a list of combinations.

Worked example 10.3

You flip a fair coin three times. Draw out the sample space, then calculate the probability of:

a landing on only tails

b the first flip being heads

c landing on tails at most once.

Solution

Understand the problem

The question is asking about different versions of the same event: flipping a fair coin three times.

Make a plan

A good way of visualising this situation would be to write out the sample space. Afterwards, we can identify which combinations in the sample space fulfil the outcome the sub-question is looking for.

Carry out the plan

Here are the possible combinations for the sample space:

$$\begin{Bmatrix} \text{TTT}, & \text{TTH}, & \text{THT}, & \text{HTT} \\ \text{HHH}, & \text{HHT}, & \text{HTH}, & \text{THH} \end{Bmatrix}$$

Hint

It is a good idea to write out sample spaces in an organised and methodical manner. That way you can be certain not to miss any combination out.

Next, we will consider each case asked in the question.

a $\begin{Bmatrix} \text{TTT}, & \text{TTH}, & \text{THT}, & \text{HTT} \\ \text{HHH}, & \text{HHT}, & \text{HTH}, & \text{THH} \end{Bmatrix}$

There is one combination with all tails, thus $P(3 \text{ tails}) = \frac{1}{8}$

Look back

This answer is reasonable since we would expect three tails to be a relatively unlikely event.

b $\begin{Bmatrix} \text{TTT}, & \text{TTH}, & \text{THT}, & \text{HTT} \\ \text{HHH}, & \text{HHT}, & \text{HTH}, & \text{THH} \end{Bmatrix}$

Four combinations start with heads, thus $P(\text{1st roll is heads}) = \frac{4}{8} = \frac{1}{2}$

Look back

Again, this is intuitively correct since the likelihood of flipping heads on any given flip is itself $\frac{1}{2}$

c $\begin{Bmatrix} \text{TTT}, & \text{TTH}, & \text{THT}, & \text{HTT} \\ \text{HHH}, & \text{HHT}, & \text{HTH}, & \text{THH} \end{Bmatrix}$

3 flips have 1 tail, and 1 has no tails

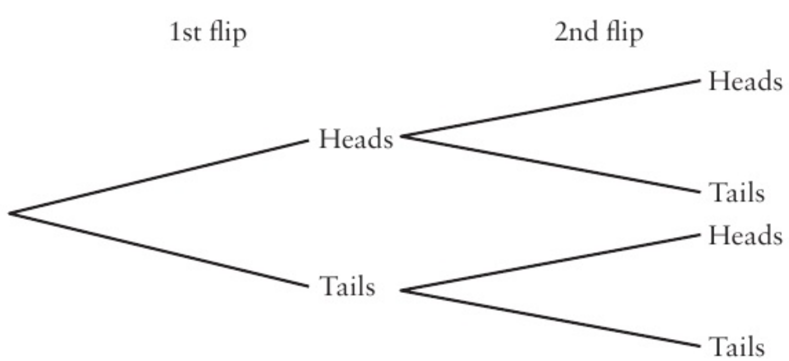
$P(\text{at most 1 tail}) = \frac{1}{2}$

Look back

Three outcomes include 2 tails, and one outcome includes 3 tails.

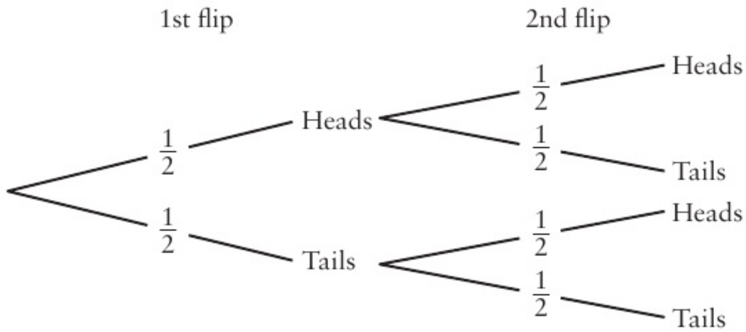
That leaves 4 outcomes, so the probability $P(T \leq 1) = \frac{1}{2}$ is correct.

The method of drawing or listing out the possible outcomes is very useful, but it can be time consuming. Another approach is a powerful tool called a tree diagram. In a tree diagram, we can map out the possible outcomes as a set of branching possibilities. For example, in the diagram below we can see the options when flipping a coin twice:



If we follow the top branch, we can see the probability of flipping one head followed by another head. Going down the branches, we can see that there are four different pathways to follow, namely {(Heads, Heads), (Heads, Tails), (Tails, Heads) and (Tails, Tails)}. Notice that this is actually the sample space of the event. This means that a tree diagram is just another way of listing the possible outcomes.

Just as in the last example, we can use the sample space to help calculate probabilities of repeated events. However, we can also use the tree diagram for this by assigning each branch its associated probability. In the situation above, each branch is equally likely to happen. Even if we get heads on the first flip, the probability of the second flip being heads remains 50/50. This gives the following final diagram:



To calculate the probability of any given event, apply the following two rules:

1 Multiply probabilities along branches.

2 Add probabilities down branches.

For example, to calculate the probability of flipping heads twice in two flips, we would move along the corresponding branch and multiply the probabilities:

$P(HH) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

Worked example 10.4

A cube has one face that is blue and the other five faces are green. You roll the cube twice and record your results.

a Draw a tree diagram representing this situation.

b Find the probability of rolling blue twice.

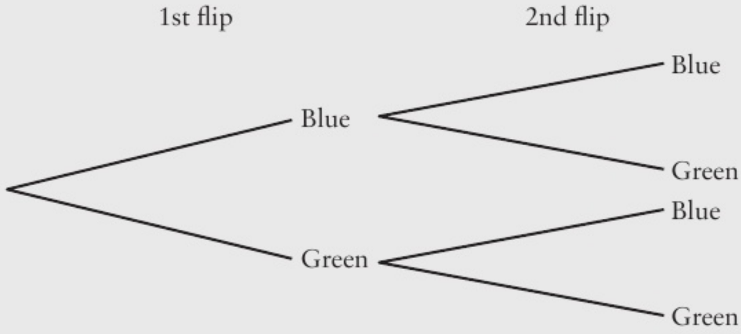
c Find the probability of rolling one blue and one green face, in any order.

Solution

This situation is describing a special dice, meaning that like any cube or dice it has six faces. Since only one face is blue, there is a 1 in 6 chance of landing on blue and a 5 in 6 chance of landing on green.

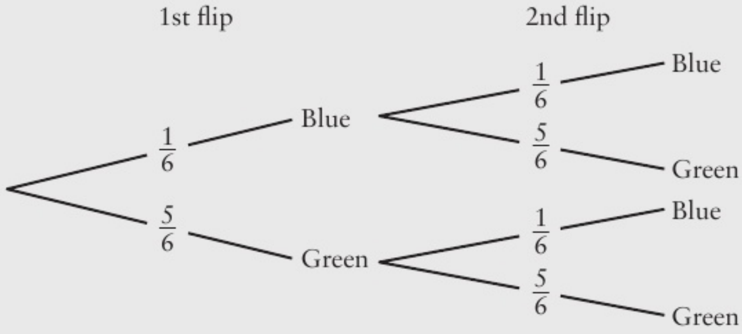
We will make a tree diagram that represents this situation for part a. We can then use this to help us solve parts b and c, by identifying the branches that match with the required outcomes.

a From the information given, we know there are only two outcomes: blue or green. Let's draw out these possible pathways first.



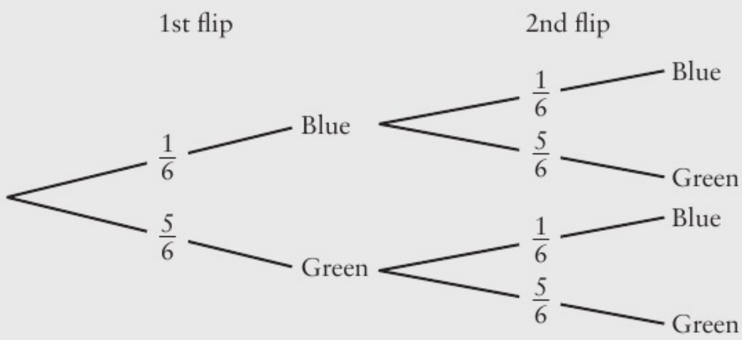
Now we can add in the probabilities to each branch:

P(Blue) $= \frac{1}{6}$ and P(Green) $= \frac{5}{6}$



Looking back, this tree diagram set-up makes sense, as it is in chronological order (first roll, second roll) and following each of the branches would lead us to the four different outcomes we can expect from rolling a cube twice.

b First, identify the appropriate branch on the tree diagram.

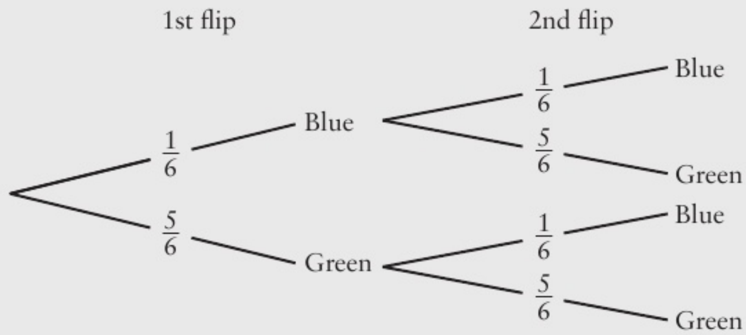


Since we are travelling along a branch, we multiply:

$$P(\text{Blue, Blue}) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

Looking back, since there is only one blue side, we can expect this to be a fairly unlikely outcome.

c Again, identify the branches that produce the desired outcome.



Multiply along branches and add between branches:

$$\begin{aligned} P(\text{B,G or G,B}) &= \frac{1}{6} \times \frac{5}{6} + \frac{5}{6} \times \frac{1}{6} \\ &= \frac{5}{36} + \frac{5}{36} = \frac{10}{36} \quad \text{Simplify if possible.} \\ &= \frac{5}{18} \end{aligned}$$

Again, since this outcome requires blue to be rolled exactly once, it is still unlikely since there is only one blue side.

Practice questions 10.1.2

1 Write out the sample space for flipping two fair coins, then use your results to calculate:

- a the probability of flipping two heads
- b the probability of flipping heads then tails
- c the probability of flipping heads and tails once each, in any order.

2 You roll two standard six-sided dice. Calculate the probability of rolling:

- a a sum of 6
- b a product of 4
- c a difference between dice of 5
- d only odd numbers.

3 You spin two four-sided spinners, each numbered 1 to 4. Write out the sample space, then calculate the probability of spinning:

- a the same number on both spinners
- b a sum of 5
- c a product greater than 7.

Hint Q2

You can use your sample space diagram from Explore 10.1 to help you in question 2.

4 You draw a card from a standard deck of cards, record its suit, then replace the card and shuffle the pack. You repeat this a second time.

a Write out the sample space to show the possible outcomes.

b Find the probability of drawing two hearts.

c Find the probability of drawing only red suits.

d Find the probability of drawing exactly one spade.

e Find the probability of drawing at least one club.

5 A box contains 6 red balls and 4 yellow balls. A person draws a ball at random, replaces it, then draws a second ball.

a Draw and label a tree diagram to describe the situation.

b Calculate the probability of drawing only red balls.

c Calculate the probability of drawing one ball of each colour.

6 A jar has 3 blue tokens, 5 purple tokens and 2 green tokens. A person picks a token at random, replaces it, then draws a second token.

Challenge Q6

a Draw and label a tree diagram to describe the situation.

b Using your tree diagram:

i calculate the probability of two purple tokens being drawn

ii calculate the probability of no purple tokens being drawn.

Reflect

Why do questions 5 and 6 ask us to replace the objects?

How would the problem change if they were not replaced?

How could we account for this in our tree diagrams?

10.2 Experimental probability

10.2.1 Applying theory to reality

British mathematician Ian Stuart famously said, ‘Mathematics is the science of patterns, and nature exploits just about every pattern there is.’ As a result, understanding the patterns of nature can help us to better understand the world around us and to make predictions.

Explore 10.2

Average summer temperatures in Athens, Greece, have been rising steadily in the last three decades. Scientists predict that there is about an even chance for a heatwave this year in July. Decide which three of the following factors might have an effect on the probability of a heatwave:

- Rainfall in June
- Rainfall in July
- Government leadership in Athens
- Car usage in Athens
- Global greenhouse emissions
- Ocean tides
- Cloud cover
- Wind patterns

Communication skills

Be prepared to justify your choices to others.

Floods are often characterised by their probability. For example, a '100-year flood' means we base our predictions on the worst flood for the last hundred years. On average, we would expect it to happen only once every 100 years. However, this does not mean that having two years of terrible floods in a row is impossible – it is just very unlikely. In fact, using our knowledge of repeated events we can calculate the probability of two 100-year floods in successive years as $\frac{1}{100} \times \frac{1}{100} = \frac{1}{10\,000} = 0.0001 = 0.01\%$. While this probability is very small, many factors, including the global climate crisis, make global patterns harder to predict, and our estimates less reliable.

We can apply the same logic when dealing with any probability experiments: the trends of the past are often a good indicator, but they are rarely perfect.

Worked example 10.5

Anastacia decides to record the temperatures in her home in Athens, Greece, to see how bad heatwaves are getting. She collects the maximum daily temperatures for the month of June and records them in the table below:

Temperature (°C)	26	27	28	29	30	31	32	33	35
Number of days with this maximum temperature	2	3	4	6	5	3	3	3	1

Using Anastacia's data, calculate the probability that a randomly chosen day in June in Athens:

a has a max temperature of exactly 29°C

b has a max temperature below 30°C

c has a max temperature of at least 30°C.

Solution

We need to calculate the probabilities based on the data provided.

Of the 30 days in June, we need to calculate how many days satisfy the temperature conditions from each question part. We can use this to find the appropriate ratio which will represent the probability.

a $P(T = 29°C) = \frac{6}{30}$ 6 of the days had a maximum temperature of 29°C.

$= \frac{1}{5}$ Simplify if possible.

Looking back, despite being the modal temperature, 6 days only represents one-fifth of the days in June.

b $P(T < 30°C) = \frac{2+3+4+6}{30}$ We add the all the temperatures below 30°C.

$= \frac{15}{30} = \frac{1}{2}$

Looking back, this makes sense as the sum of days is half the days in June, namely 15.

c $P(T \geq 30°C) = 1 - \frac{1}{2}$ This is the complement to the answer for part b.

$= \frac{1}{2}$

Looking back, we could check our answer by finding the sum of days with a maximum temperature of 30°C or more. This would give $5 + 3 + 3 + 3 + 1 = 15$, which still represents half the days in the month of June, confirming our answer.

Reminder

A complement is the opposite of a probability event. Remember that $P(X) + P(X') = 1$, where X' is the complement of X.

Reflect

How reliable are the answers to Worked example 10.5 at predicting future years' maximum temperatures in Athens? Give at least two specific reasons to support your answer.

Practice questions 10.2

1 Experts predict more flooding in Karachi, Bangladesh, with a 1-in-10-year flood being recorded in 2020.

  a What is the probability of two successive 1-in-10-year floods in a row?

  b What about three years in a row?

  c What other factors might influence this estimate?

2 Bush fires are becoming more and more common in Australia. Experts warn that, on average, 70% of summers in the 2020s will see uncontrollable bush fires.

a What happens in the other 30% of summers?

b Construct a tree diagram to visualise two summers of Australian fires.

c Calculate the probability of two successive years of uncontrolled bush fires.

d Use your knowledge of repeated events to estimate the probability of three successive years of uncontrolled bush fires.

e How accurate do you think your predictions in parts c and d are? Justify your answer.

Fact

Microplastics and tiny particles of plastic are damaging for our environment, and particularly our oceans. They are leached, or emitted, from many plastic products, including synthetic clothing items.

3 An analysis of sports garments found that a certain number of clothing items leached microplastics when they were washed. Here is the data the analysis gathered:

Leaching	None	On first wash only	Through garment lifetime
Frequency	16	8	10

a What is the probability that a garment from this analysis, selected at random, leaches any microplastics?

b What is the probability it will leach microplastics for its entire lifetime?

c Can we extrapolate this data for other clothing items? Justify your answer.

4 Val is looking at the rainfall in his hometown for each month during a year. He collects the following data:

Month	Jan	Feb	Mar	Apr	May	Jun
Total monthly rainfall (mm)	21	28	39	39	61	64

Month	Jul	Aug	Sep	Oct	Nov	Dec
Total monthly rainfall (mm)	67	67	50	33	43	31

a Using the data, calculate the probability that a randomly chosen month:

i has a rainfall of exactly 39 mm

ii has less than 40 mm of rain

iii has more than 40 mm of rain.

b How reliable are your answers to part a at predicting future years' rainfalls? Give at least two specific reasons to support your answer.

5 Savina installed photovoltaic cells to help supplement her household electricity and reduce her family's environmental footprint. As a nice bonus, the photovoltaic system keeps track of how many hours a day her household uses 'only' solar power, and is therefore off-grid. Savina collects these data for two weeks and puts them in the table below:

Hours off grid per day (h)	$0 \le h < 1$	$1 \le h < 2$	$2 \le h < 3$	$3 \le h < 4$	$4 \le h < 5$	$5 \le h$
Number of days	4	4	2	3	0	1

a Using the data, calculate the probability that a randomly chosen day:

i has less than one hour off-grid

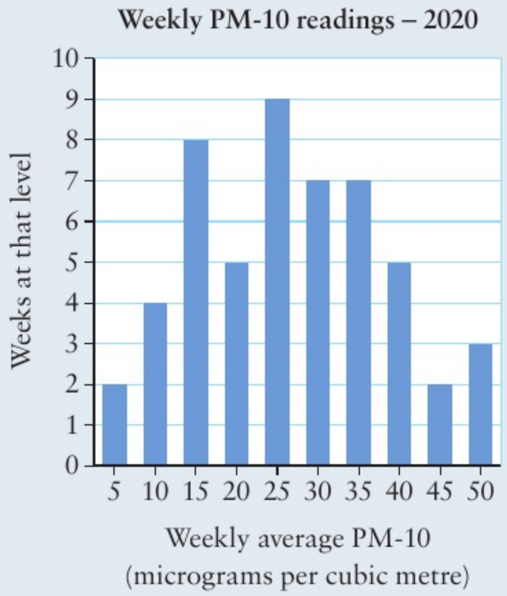
ii has at least two hours off grid.

b Can Savina use this data to predict future weeks? Why or why not?

Fact Q5

Photovoltaic cells are a type of solar panel which generate electrical power.

6 Local pollution and poor air quality is a concern in many cities as they cause environmental and health problems. A common measure of local air quality is PM-10, short for 'particulate matter under 10 micrometres'. A school in Singapore decided to test for PM-10 every week in 2020. They rounded their readings to the nearest 5. Here are their results:



Challenge Q6

Fact Q6

PM-10 are microscopic particles of dust, dirt and debris, often created by cars or factory processes, which can be harmful when breathed in. Governments often record the amounts of PM-10 in the local air and use it to help measure local air pollution levels.

Hint Q6a

Remember that there are 52 weeks in a year.

a Using the data, calculate the probability that a randomly chosen week:

  i has PM-10 levels of exactly 30

  ii has PM-10 levels of at most 20 micrograms per cubic metre.

b PM-10 levels are said to be harmful above 40 micrograms per cubic metre. What was the likelihood of a randomly chosen week having harmful PM-10 levels?

c What might be some sources of unreliability with the data collected by the school?

Investigation 10.2

Theoretical vs experimental probability

In this chapter you have learned about a lot of different probability experiments, from rolling dice to flipping coins, drawing names out of hats, or choosing marbles or cards. Your task is to design a probability experiment using the materials you have available. Using your design, you will choose at least two different probabilities to calculate.

Read through the task-specific criteria before beginning. Show all necessary working, and use appropriate terminology and symbols wherever required.

Part A: Theoretical probability calculations

1 Outline your experiment set-up. This should include your materials and the method for each trial of your experiment.

2 Express your two (or more) chosen probabilities in words.

3 Express your chosen probabilities in symbols.

4 List the possible outcomes for your chosen probabilities.

5 Calculate the probability of each of your chosen probabilities. Make sure to show all relevant steps to ensure clear communication. Answers should be left as a simplest fraction or rounded to 2 decimal places.

Here is an example of how you might want to lay out your work:

Experiment set-up	Probability 1	Probability 2
Materials: Method:	In words: In symbols: List of relevant possible outcomes: Calculation of probability:	In words: In symbols: List of relevant possible outcomes: Calculation of probability:

Part B: Experimental probability calculations

You are now ready to test your experiment! You will do this by repeating your experiment 100 times and recording the results. Follow the instructions below:

1 Construct a table to collect your data, ensuring you leave enough space to stay organised and clear. Be sure to leave space for any information that will help you with your final probability calculations.

2 Run your experiment 100 times, recording all relevant data.

3 Organise your data, paying special attention to trials that fulfilled the conditions of your chosen probabilities.

4 Calculate your experimental probabilities.

5 Complete the table below, comparing your experimental probabilities to your theoretical results from Part A.

Probability statement	Theoretical probability	Experimental probability
P( )		
P( )		

Reflect

Look back at your investigation.

How did your theoretical and experimental results compare?

Were there any surprising/unexpected results? What may have caused these?

What general conclusions can you make about probability when it is applied to real life?

If you could go back to redesign or retest your experiment, what improvements would you make? Why?

Self assessment

- I can classify outcomes as impossible, unlikely, equally likely, likely or certain.
- I can interpret probability symbols and notation.
- I can list sample spaces.
- I can calculate the probability of a simple event.
- I can identify the complement of an event.
- I can calculate the probability of an event's complement.
- I can calculate the probability of repeated events with constant probabilities.
- I can draw and use tree diagrams.
- I can estimate probabilities using data.

Check your knowledge questions

1 Organise the following outcomes as impossible, very unlikely, unlikely, equally likely, likely, very likely or certain.

a A flipped coin landing on tails 5 times in a row.

b Drawing a number card from a standard deck of cards.

c A standard six-sided dice landing on a factor of 30.

2 Describe the following probability statements in words:

a $P(\text{even}) = \frac{1}{2}$

b $P(X > 7) = 0.02$

c $P(\text{blue}) = \frac{8}{9}$

3 A box of pralines has marzipan and nougat flavours. There are twice as many marzipan pralines as there are nougat ones.

  a Write down the relationship between the events 'choose a nougat praline' and 'choose a marzipan praline'.

  b Calculate the probability of getting a marzipan praline.

  c Hence, write down the probability of getting a nougat praline.

4 You roll a 20-sided dice.

  a Write down the sample space.

  b Using the sample space, calculate the probability of rolling:

    i a multiple of 3

    ii a factor of 20

    iii a prime number

    iv a number divisible by 7.

5 A jar has 12 counters inside: 6 are yellow, 4 are green and the rest are orange.

  a Write down how many outcomes are in the sample space.

  b When drawing one at random, calculate the probability that the counter is:

    i yellow

    ii green

    iii orange.

6 You flip a coin, and simultaneously roll a dice.

  a Draw out the sample space of this probability experiment.

  b Using your sample space, calculate the probability of:

    i landing on heads, with any number combination

    ii having an even number on the dice

    iii landing on tails and rolling a prime number.

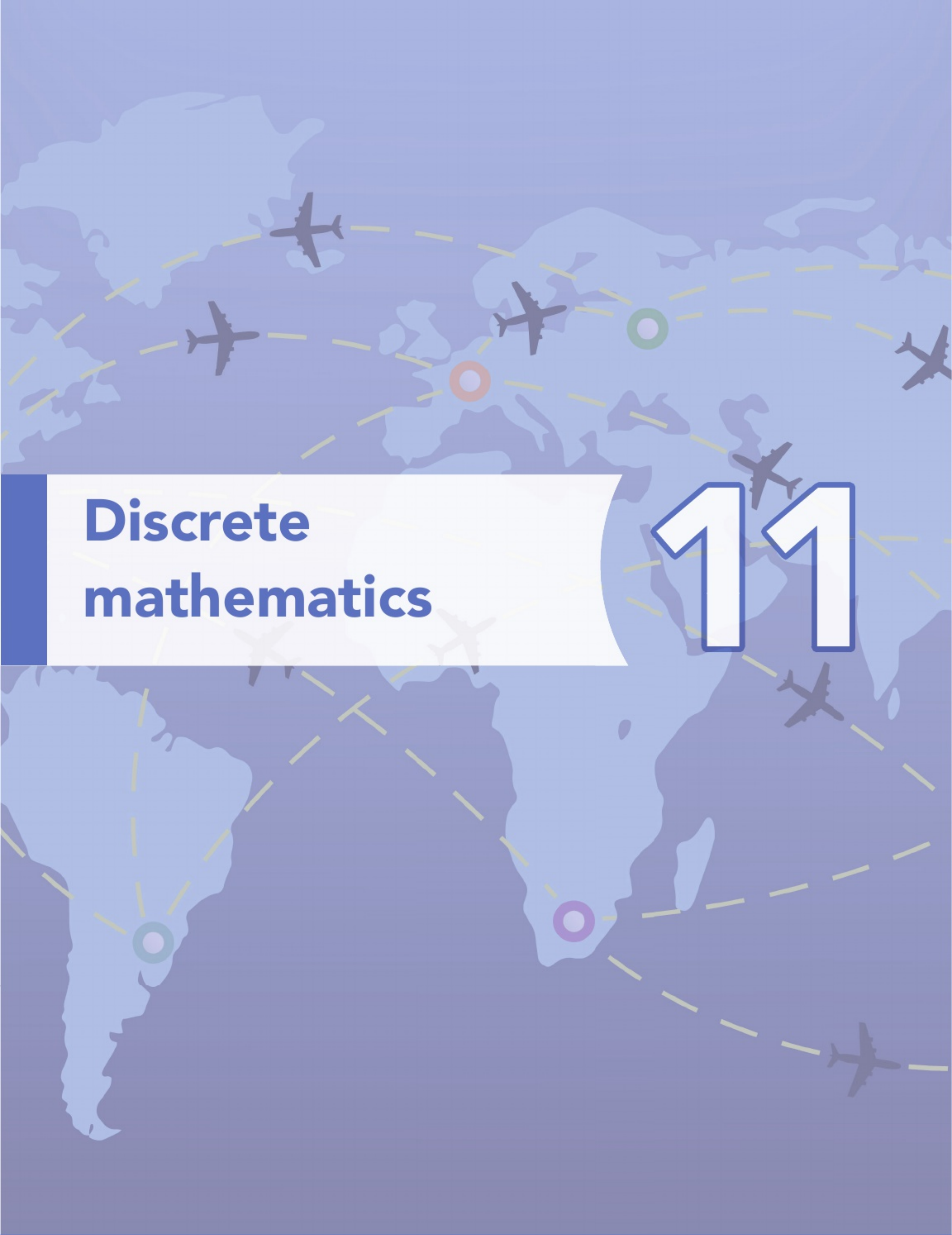
7 You roll a 12-sided dice twice and record whether the outcomes are prime or not.

a Draw and label a tree diagram to describe the given probability experiment.

b Calculate the probability of rolling two prime numbers in a row.

c Calculate the probability of rolling a prime number and then a non-prime number, in that order.

d Calculate the probability of rolling a prime number and a non-prime number, in any order.



11 Discrete mathematics

11 Discrete mathematics

KEY CONCEPT

Form

RELATED CONCEPTS

Models, Representation, Generalisation

GLOBAL CONTEXT

Scientific and technical innovation

Statement of inquiry

Representing real-life situations by using innovative graphical forms supports us with complex problem-solving.

Factual

- How can complete graphs be used to solve real-world problems?

Conceptual

- How do we know whether a graph has a circuit?

Debatable

- Can we always be sure that minimum spanning trees are really minimum?

Do you recall?

1 List all possible subsets of the set $\{1, 3, 5\}$

2 You will be required solve problems by yourself. There are pointers given by Pólya to help in solving problems. The following is a list of pointers, two of which are not Pólya's. Which ones are not Pólya's?

- **a** Look for a simpler problem
- **b** Use the quadratic equation
- **c** Make a table
- **d** Work backwards
- **e** Make educated guesses
- **f** Look for patterns

3 A map has the scale 1 : 100 000. If two cities on the map are 5 cm apart, how far apart are they in kilometres?

11.1 Introduction to graphs

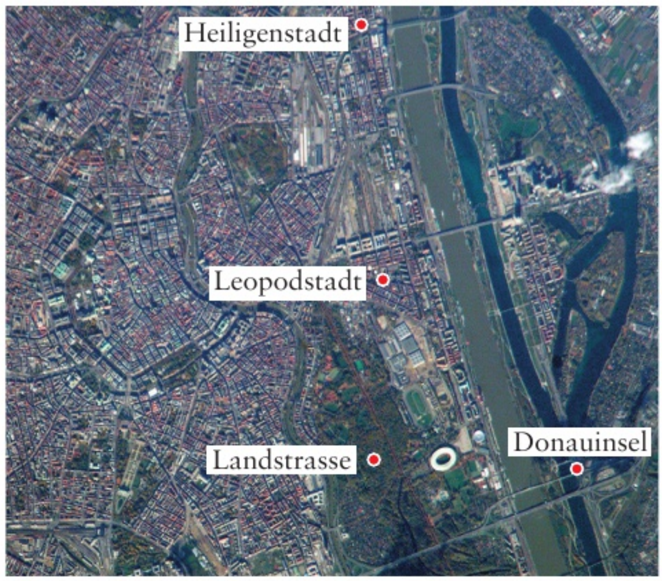
> **Reminder**
>
> In Chapter 5, we used 'graph' to describe the straight-line graphs in the Cartesian plane. Do you remember any other types of graphs we have already used?

> **Hint**
>
> The plural of vertex is **vertices**.

You are already familiar with the word 'graph' in mathematics.

In this chapter we will look at a different type of graph. A **graph** is a collection of nodes, some or all of which are connected to each other by line segments. Each node is called a **vertex** and each line segment is called an **edge**.

Examples include graphs representing railway or subway networks, such as the Vienna Metro map shown below left. Other examples of graphs are oil pipelines, electricity nets, flight paths of airplanes, postal delivery routes, etc.



Explore 11.1

Compare the two maps of Vienna and answer the following questions.

a What similarities are there between the two maps?

b What are the biggest differences?

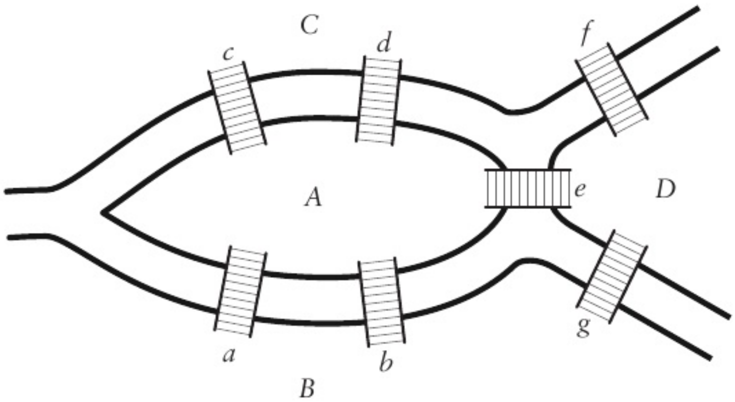
c What are the main advantages of the metro map?

d Assume you need to go from Donauinsel to Heiligenstadt. Which map is more helpful?

e Do the two locations have the same positions in both maps?

In this chapter we will study **graph theory**. This discipline was pioneered by the famous Swiss mathematician Leonhard Euler (1707–1783).

Euler was intrigued by the question of whether a route existed that would traverse each of the seven bridges of Konigsberg exactly once. In demonstrating that the answer is no, he laid the foundation for graph theory.



This problem could be formulated as follows:

> 'Is it possible to cross all 7 bridges once, and return to the place where you started?'

Explore 11.2

In order to answer this question, follow these steps:

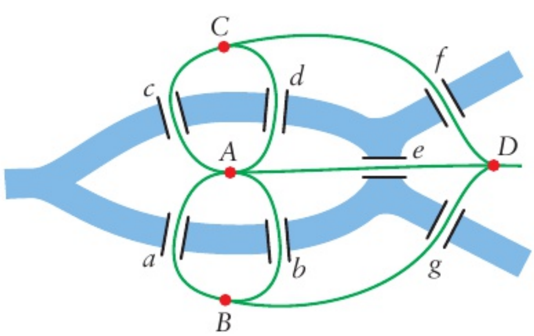
Trace the figure on a piece of paper. Can you cross all 7 bridges as described?

If the answer is yes, show exactly how. If the answer is no, explain why not.

The key to solving the problem lay in developing a model, with its own notation and concepts.

The first thing Euler did was to distinguish between the bridges and the landmasses they connect. He represented the bridges by **edges**. He represented the landmasses by **vertices**.

Here is the resulting representation:



Explore 11.3

Copy the representation on paper, large enough to draw on.

Remove one bridge joining A to either B, C or D. Try to cross all the bridges and return to where you started. What do you notice?

Now, remove the bridge between A and D and add a bridge between C and B. Try to cross all the bridges and return to where you started. What do you notice?

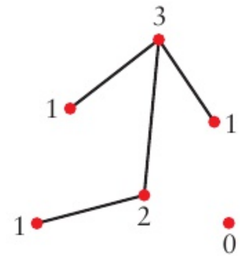
Make your own changes in the number of bridges between the different landmasses. Which arrangement will make a round-trip possible, but keep the number of bridges to a minimum?

Can you generalise what you observed?

The Königsberg bridges graph is an example of a **connected** graph, because there are no landmasses that cannot be reached by a bridge. In a connected graph, each vertex is connected to at least one other vertex.

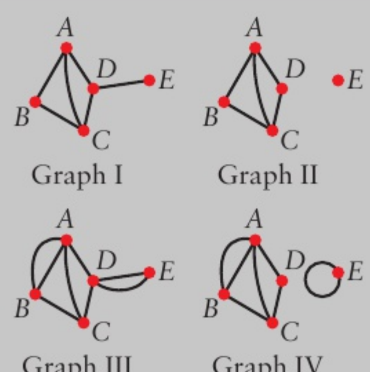
The number of edges connecting a vertex to other vertices is referred to as the **degree** of the vertex.

The diagram shows a simple graph with the degree of each vertex labelled.



Worked example 11.1

Graphs can be connected or disconnected, and simple or not. Look at the four diagrams:



Graph I is simple connected.

Graph II is simple disconnected.

Graph III is not simple but connected.

Graph IV is not simple and disconnected.

a Can you propose a definition for a simple graph?

b Can you propose a definition for a connected graph?

Using your definition for a simple graph, do you think that the Königsberg bridges graph is a simple graph? Explain your answer.

Fact

Any two vertices that are connected by one edge or more are said to be **adjacent vertices**.

Solution

We need to find a definition of connected and simple graphs by examining the features of each graph.

There are two simple graphs and two connected ones. We'll try to see what is common for each pair.

Looking at simple graphs I and II, we notice that if any two vertices are connected, the connection is with only one edge. This is confirmed by noticing in graphs III and IV that the non-simple graphs have more than one edge connecting some of their vertices. In addition, graph IV has a loop at E.

By examining II and IV, we notice that they each have one isolated vertex. That is, a vertex from which we cannot reach every other vertex of the graph. We now have all information to make our definitions:

a In a **connected graph**, every vertex can be reached from another vertex by following the edges.

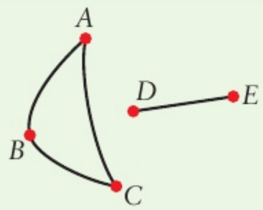
b A **simple graph** has at most one edge between vertices, and no loops.

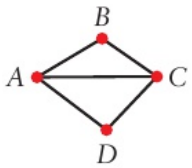
Following these definitions, you can say that the graph of the Königsberg bridges is connected, but not simple, as there are two edges between A and C, for instance.

Looking back, the definitions we found rely on some specific examples of graphs. A solid understanding may require that we research the concepts further.

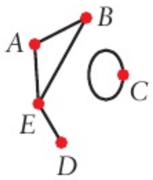
Reflect

What if a graph does not have a single isolated vertex but has subsets of its vertices connected within themselves, but separated from the rest of the graph's vertices? Here is a sample.

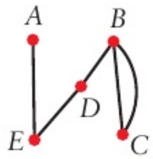




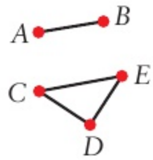
Graph I



Graph II



Graph III



Graph IV

Hint Q2

When you draw a graph, start by plotting the vertices and labelling them *A*, *B*, etc. Space them out a bit in a roughly circular fashion. Then add in the edges. Note that in this question you are also looking for graphs that are not simple.

Hint Q3

A graph that can be rotated or reflected to become another graph is not considered to be different.

Practice questions 11.1

1 For each of these graphs on the left, state:
   a the number of edges
   b the number of vertices
   c whether or not the graph is connected
   d whether or not the graph is simple.

2 Draw a graph with 3 vertices and 3 edges. How many different graphs can you find? How many of them are simple and connected?

3 Draw a simple graph with 4 vertices and 4 edges. How many different graphs can you find now?

4 Draw a simple graph with 4 vertices and 6 edges. How many different graphs can you find? What do you notice?

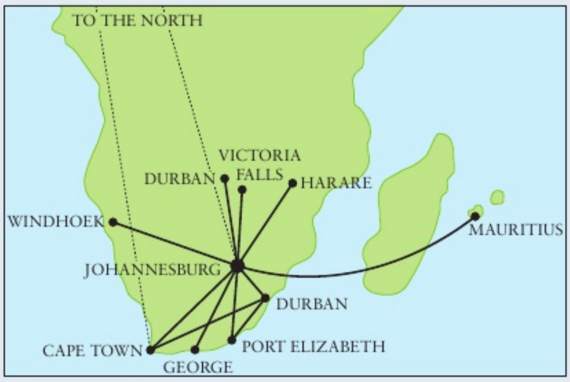
5 In the map of the Vienna metro system:
   a What does each vertex represent?
   b What does each edge represent?
   c What is the degree of Karlsplatz?

   An interchange is where you can change from one line to another.

   d How many interchanges are there on the section of the Vienna metro shown at the start of this section? What are their degrees?
   e What is true about vertices with degree 1?
   f Why can there be no vertex in this network with degree 0?

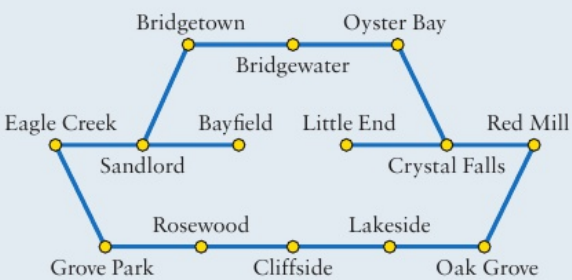
6 Compare the geographic map of Vienna to the map of its metro system.
   a Describe one advantage of the metro map.
   b Suggest how you could improve the map of the metro system.

7 a How many vertices are there in the diagram on the next page showing an airline's route network?
   b Why is it useful for customers to have a map that is to scale?



8 The diagram shows the subway system in Rome. Which stations do you expect to the the busiest and why?

9 Below is a rail network for a small town and a simple map of the same town with the actual positions of the stations.



a Try to match the vertices on the network with the stations on the map.

b What information does the map tell you that the network doesn't? (Assume you know the scale used in drawing the map.) What could be included on the network to supply this information?

Research skills

10 Find the geographic and metro or railroad maps from the capital of your home country, or that of your school. Compare the two maps and comment on the way the metro (or rail) map simplifies the geographic reality by looking in more detail.

11.2 Subgraphs, complete graphs and trees

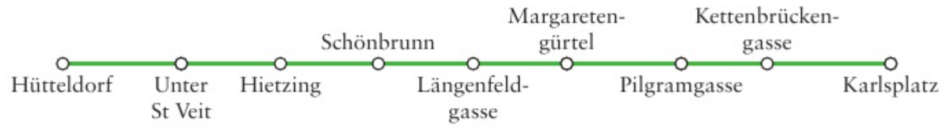
A graph can be described as consisting of a set of vertices and a set of edges. A **subgraph** will then feature some or all of the vertices, and some or all of the edges of the original graph. You could say that the set of elements of the subgraph is a **subset** of the elements of the original graph.

Connections

Sets and subsets

A subgraph is to a graph what a subset is to a set: it has some of the same elements.

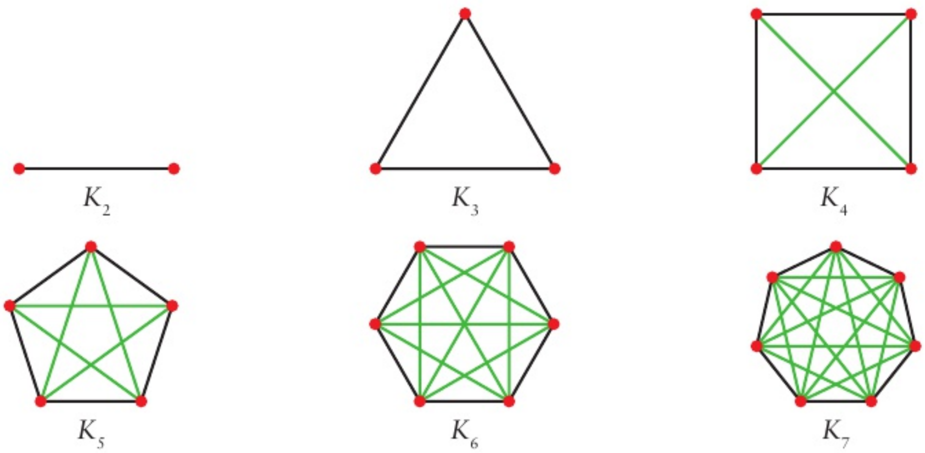
Below is an example of a subgraph of the Vienna metro graph. It represents part of the U_4 line within the Vienna network.



It has all the vertices and edges that appear on the original, so it is a subgraph.

In Practice questions 11.1 you should have found that all the graphs in question 3 are subgraphs of the graph in question 4. The graph in question 4 is called a **complete graph**. A complete graph has the maximum number of edges that a simple connected graph can have. You could compare it to the universal set, as it contains all the elements.

Here are the first six complete graphs:



Explore 11.4

Refer to the first six complete graphs in the diagram above. Collect your findings in a table.

How many edges does each of $K_3, K_4, \cdots, K_7$ have?

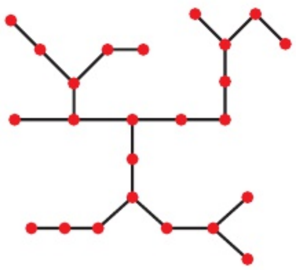
Can you predict how many edges K_8 and K_9 will have?

Can you predict how many edges K_n (a complete graph with n vertices) has?

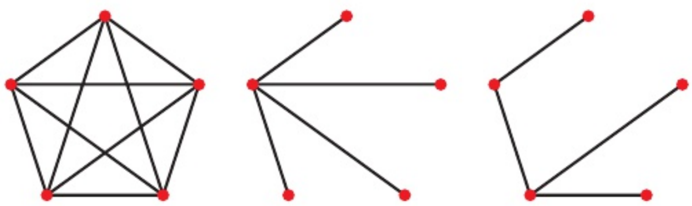
A special type of connected graph where there is only one way of getting from one vertex to another is called a **tree**. A tree is a simple graph.

There are no cycles in a tree. Thus, you cannot start at one vertex and return to it without retracing the same edges.

Here is an example of a tree:



A **spanning tree** of a graph is a subgraph that is a tree and contains *all* the vertices of the graph. Here is a graph and two of its spanning trees:



Fact

A **path** is a set of edges that join a sequence of vertices.

A **cycle** is a path that starts and ends at the same vertex.

Worked example 11.2

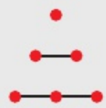
How many edges does a tree of n vertices have?

Solution

We need to find a formula that gives the number of edges for a tree with any number of vertices.

In order to find this formula, we need to find a pattern connecting the number of vertices to the number of edges in the corresponding tree. Start with the simplest situation: 1 vertex and 0 edges. Then draw trees with increasingly more edges and vertices and record the findings in a table. Once we have these values, we can spot the pattern and find the formula.

Let's start with 1 vertex. Clearly there will be 0 edges, as there are no other vertices to connect it to. The next situation will consist of 2 vertices and 1 edge, then 3 vertices and 2 edges, as illustrated in the diagram. Recording this in a table we get the following:

1:
2:
3:


Vertices	Edges
1	0
2	1
3	2

Using these results, we might suggest that the number of edges is always one less than the number of vertices.
We call that a **conjecture**. To check whether this conjecture is correct, we draw a diagram:

4:


We see that the pattern holds, so we are confident that the pattern is correct. We can now suggest the formula for the general case: a tree with n vertices has $n - 1$ edges.

In this example we found the requested result by following these steps: observation and data collection, organising data, spotting a pattern, making a conjecture, testing the conjecture and, finally, reaching the requested result. This sequence of steps is useful to remember and you are advised to use it in question 9 below.

Connections

A conjecture in mathematics is similar to a hypothesis in science. Checking that a conjecture is true for a number of cases does not prove that it is true for all cases. To do this you need to know about formal methods of proof, which you will learn later in your studies.

Reflect

Did you manage to solve the problem posed in Explore 11.4? If so, did you follow the same steps as outlined in the example above? If not, try again using these steps.

Practice questions 11.2

1 Can you make a tree with:

- **a** more edges than vertices
- **b** the same number of edges and vertices?

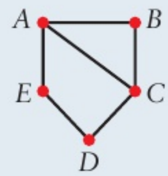
2 When we compare international airline networks and railways, which do you think is more likely to be represented by a tree? Explain your answer.

3 From the graph shown, draw a subgraph that:

a is a not a tree

b has more vertices than edges

c is disconnected.



4 Look at the graph in question 4. How many different spanning trees can you draw? Show your working.

5 Look again at the graph in question 4. Which edges do you need to add to make it a complete graph?

6 Draw the three different possible spanning trees from K_4.

7 Constellations of stars in the night sky are represented by trees. Can you find two constellations that can be represented by the same tree? You may have to move the stars around a bit.

8 Imagine you are in a group of 4 students. You need to team up in pairs to do a certain task. How may pairs are possible? How many if you were with 5 students, or 6?

Imagine you did the same with everyone in your class. How many pairs would be possible?

Hint Q9

Make a table containing the number of students and the corresponding number of pairs. Then follow the steps outlined in the previous worked example.

Challenge Q9

11.3 Euler circuit and Euler trail

In the previous section, we looked at different aspects of graphs that help us better understand how graphs work. In this section, we return to the problem of the Königsberg bridges.

To recollect, this was a problem Euler was asked to solve:

> 'Is it possible to cross all 7 bridges once, and return to the place where you started?'

We are now going to discover how this problem is related to the number of bridges (edges) connecting the different vertices (landmasses); that is, the degrees of the vertices.

We now reformulate the Königsberg bridges problem in terms of graph theory:

> 'Is it possible to travel along each edge once and return to the vertex where you started?'

Hint

You may not find it easy to understand this technical language, but we could simplify it as follows:

'Can you trace each of the edges of the graph once without lifting your pencil off the paper, starting and finishing in the same place?'

> **Fact**
>
> An Euler trail is also called an **Euler path**.

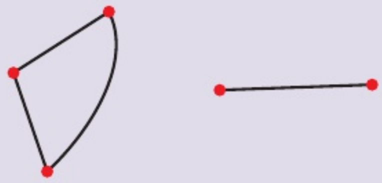
If it is possible to trace each edge once, not necessarily finishing where you started, we say that the graph has an **Euler trail**. An Euler trail is a walk through the graph that uses every edge **exactly** once.

In addition, if your starting vertex is the same as your ending vertex, we call it an **Euler circuit.**

In order to solve Euler's problem, we are going to explore some simpler cases first.

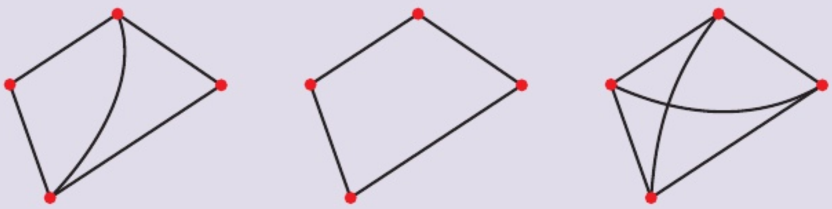
Explore 11.5

Here are two graphs. One has an Euler circuit and the other has an Euler trail only. Can you tell which is which, and argue why?



Graph I Graph II

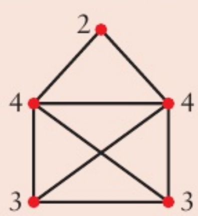
Here are three graphs. One has an Euler circuit, one has an Euler trail only, and one does not have either. Can you tell which is which, and argue why?

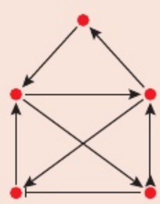


Graph I Graph II Graph III

Investigation 11.1

For each of the graphs shown in the table below, trace the edges of the graph without lifting your pencil and without tracing an edge twice. Here is an example of a graph with an Euler trail and how it can be drawn:



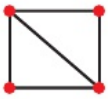
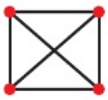
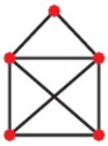
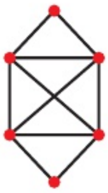


In addition, you need to write down the degrees of each of the vertices of each graph, as shown in the example above.

Euler made a distinction between vertices of odd and even degree. Can you think why?

Now record your information in the table below.

Complete the table for the given graphs and indicate which ones have Euler trails and which have Euler circuits.

Graph	Number of vertices with even degree	Number of vertices with odd degree	Euler trail?	Euler circuit?
				
				
				
				
				
				
				
				

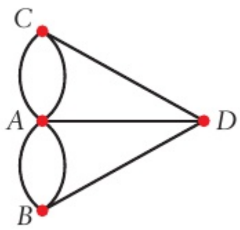
Hint

Think about what it means for a landmass to have an odd or even number of bridges connecting to it.

Can you find a pattern in the degree of the vertices in the graphs, and whether or not it has an Euler circuit?

Can you find a pattern in the degree of the vertices in the graphs, and whether or not it has an Euler trail?

Test your conjecture by drawing a couple of graphs of your own to see if it works.

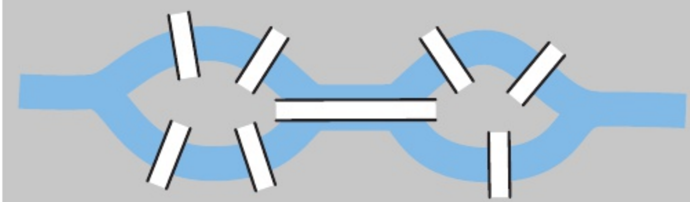


Now it's time to return to the Königsberg bridges. Look at the graph on the left and write down the degrees of the vertices. Copy and complete the following statement, selecting the correct conclusion:

> 'An Euler circuit is/is not possible in Königsberg, because…'

Worked example 11.3

Consider the following imaginary situation in the fictitious town of Eulerville:



Jan claims the following:

> 'It is possible to walk each of the bridges once, but you cannot end where you started.'

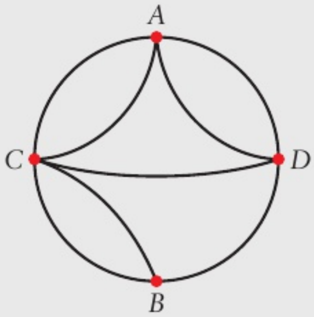
Do you agree? Explain your reasoning using graphs.

Solution

The question is asking if there is an Euler trail possible in Eulerville.

The plan is to draw a graph representing the situation with landmasses as vertices and bridges as edges. We also need to label the landmasses to make our task simpler.

Label the two outer regions as A and B and the two islands as C and D.



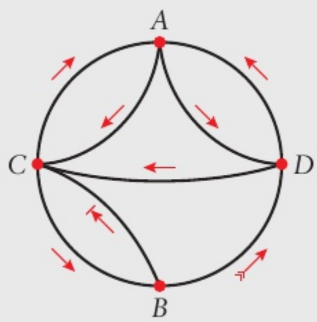
You may have gathered that for an Euler circuit to exist, every vertex must have an even number of edges connected to it. When we visit a vertex, an edge is required to go in and an edge to go out so that we do not retrace any edge.

A and D do not present a problem here because each one of them has a degree of 4. If we start at B, we won't be able to end there (after traversing every edge exactly once). After using one edge to leave the vertex, we will be left with 2 edges stemming from it. One of the two edges left could be used for returning to the vertex, the other one for leaving. So we return, then leave. Thus, we cannot end at B. The only way to use up all the edges is to use the last one by leaving this vertex. A similar argument holds for C.

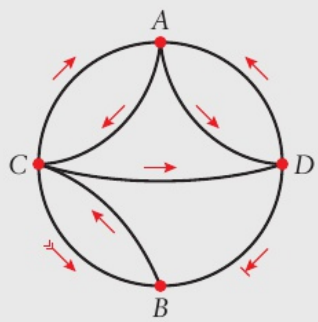
On the other hand, if we do not start a path at B, then we will eventually get stuck at that vertex. The path will use pairs of edges incident to the vertex to arrive and leave again. Eventually, all but one of these edges will be used up, leaving only an edge to arrive by, and none to leave again. Thus, we cannot start at A or D. An Euler circuit is not possible.

However, it is possible to have an Euler trail under the condition that we start at B and end at C, or start at C and end at B. Here are two scenarios:

BDADCACBC



CBCACDADB



Looking back at the physical setup, our solutions make sense since the first one uses the bridge from B to D, then uses the two bridges back and forth between D and A, crosses the only bridge from D to C, back and forth between C and A, and finally back and forth between C and B ending at C.

A similar description applies to the other trail.

Reflect

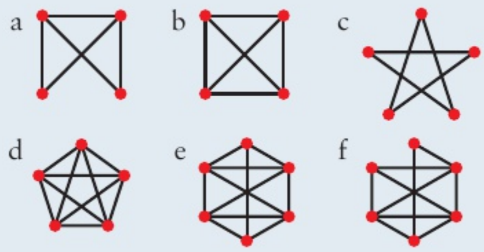
In this section, you have discovered that graphs can have an Euler circuit only if all vertices are of even degree and that this is almost possible when there are two vertices of odd degree. In that case, there is an Euler trail.

Explain in your own words why these conditions are necessary for a circuit or a trail. Use the Königsberg example and use the language of bridges and land masses.

Practice questions 11.3

1 Write down the degree of each vertex in these six graphs.

2 Which of these graphs have an Euler trail or circuit? Check your answer by tracing it with your pencil.



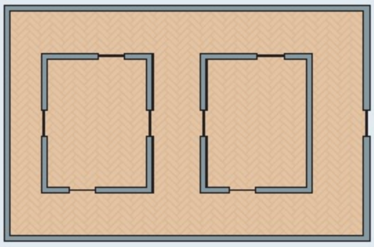
3 For the same graphs, add the degrees of the vertices in each graph and compare it to the number of edges. What do you notice? Can you explain why this must be the case?

4 Make up a connected simple graph of your own with the following properties:

a five vertices and an Euler circuit

b six vertices and an Euler trail

5 The diagram shows the floorplan of a museum.



a Explain why it is not possible to go around the museum, visit each room and exit, having used every door only once.

b Suggest how you could make this possible by adding or removing one door.

6 Consider the floorplan of your own house. Is it possible to visit each room, go through each door only once and exit again? Explain why this would not be possible in most houses.

7 Imagine you are an architect and you are asked to create a floorplan that makes it possible to go through each door once and exit through the same door you entered. Given that you need to have four rooms, make a design of such a floor plan.

8 If you were to make a design with the same specifications as the previous questions, but now with seven rooms, what is the minimum number of doors you would need?

9 As in question 8, but now with n rooms, what is the minimum number of doors you would need?

10 Jakob claims that all complete graphs contain either an Euler circuit or a trail. Do you think he is correct?

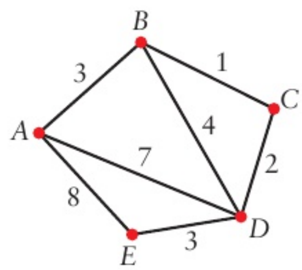
Can you make a claim of your own based on what you found?

Hint Q10

Remember that in a complete graph each vertex has the same degree.

11.4 Weighted graphs and problem solving

In this section we look at solving problems with weighted graphs. Remember that a graph is a set of vertices and edges, where each edge signifies a link between two vertices. A **weighted graph** is a graph where each edge has a numerical value called a **weight**. Below is an example of a weighted graph:



The graph consists of vertices and edges as before. However, each edge has some weight. For example, edge AD has weight 7, edge BD has weight 4, and so on.

Weighted graphs are often used to model real items and processes. For example, the graph above can be considered as a map, where the vertices are towns, and the edges are roads. The weight of each edge is the distance between two towns.

Fact

A weighted graph is often called a **network**.

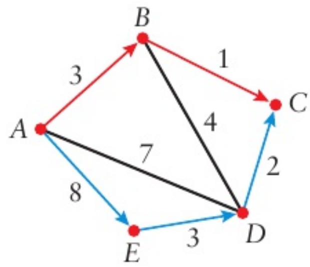
Explore 11.6

Look at the diagram of the weighted graph and consider it as a map between cities. Can you suggest what is the shortest distance to travel between towns A and D?

A **path** in a weighted graph is the same as in an unweighted one. The only difference is that in a weighted graph, a path has a **weight** that is equal to the **sum of the edge weights** in the path. For example, consider the path A to E to D to C (shown by blue arrows):

Hint

We will use the words trail and path to mean the same thing.



The path consists of three edges. The total weight of the path is the sum of the weights of the edges:

$$8 + 3 + 2 = 13$$

You may notice that there are other paths from vertex A to vertex C, one of which is shown by red arrows.

The weight of this path is $3 + 1 = 4$, which is less than the weight of the blue path.

The problem of finding the shortest path (a path with the minimum weight) between two vertices in a graph often arises in programming practice. There are several algorithms for solving the shortest path problem. We will consider one them in more detail next.

We are going to look at a famous type of problem linked to weighted graphs. In this type of problem, the weights represent distances, cost, time, and so on. The problem we need to solve is to find the shortest trail to visit all vertices. We call this the **minimum spanning tree**, because we are looking for a spanning tree with minimum weight.

We also use weighted graphs to find trails/paths with minimum weight between two specific vertices.

Explore 11.7

Look at the diagram at the top of the page again. Assume that the numbers on the edges represent the distances between towns in kilometres.

We need to set up a bus route that covers all these cities. To save cost, we need the shortest network that connects all the towns. Can you set up the bus route?

Minimum spanning tree

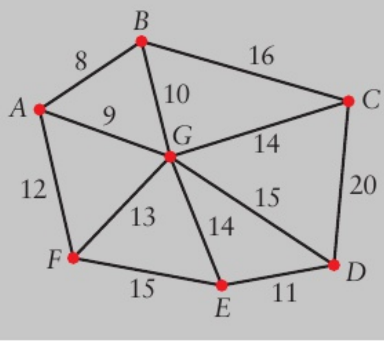
A spanning tree having the smallest weight among all other spanning trees is called a **minimum spanning tree**.

When graphs are relatively small and simple, like the one at the beginning of this section, we can find the solution by inspection. We start by using edges with minimum weight then build the tree by adding the smallest possible edges connected to it until we include all vertices. Such plans are called **algorithms**. The process is illustrated with the following example.

Worked example 11.4

The diagram shows the cost (in thousands of dollars) of connecting a number of farmhouses to the electricity network.

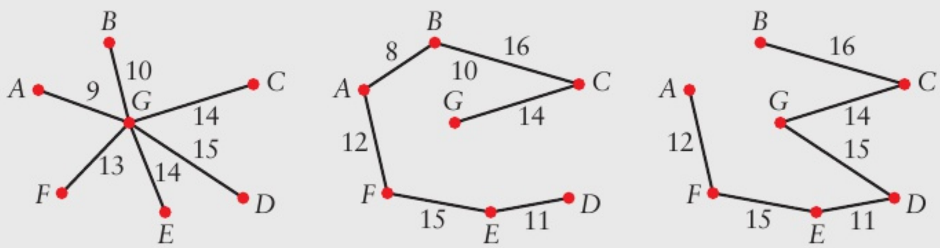
Find the cheapest solution for connecting all the farmhouses to the electricity network.



Solution

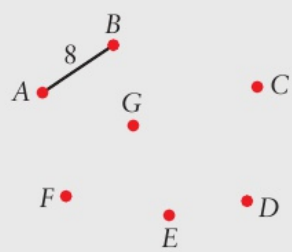
In order to find the cheapest solution for connecting all of the farmhouses to the network, we could simply try different trees until we find the cheapest one. This could take a lot of time, as there are many spanning trees possible. Some examples are shown here. The total costs of these spanning trees are 75, 76 and 83 respectively.

Can you work out how many different spanning trees there are?



Instead, we could be more systematic in our approach. Start at any vertex and choose the edge with the lowest value. Now we have two vertices in our network and we will choose the next edge with the lowest value. Continue in this way until we have all vertices in our network.

1 Start with any vertex, say A, and join it to the closest vertex; in this case B.

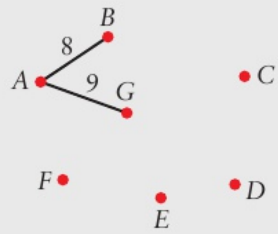


Fact

Various algorithms have been designed that allow computers to find the solution when dealing with graphs with large numbers of vertices. You will not be asked to solve those, but the underlying logic is not really different from what you will use in the simpler graphs.

Challenge

2 A and B are now both part of a network. We look for the next lowest edge connecting either vertex A or B to the other vertices. This is the edge connecting A to G, which gives us the following network:



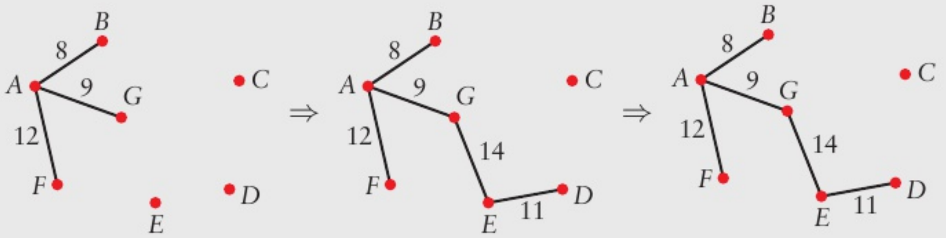
3 Now do the same from the new edges. The closest unconnected vertex from the edges we now have is F.

We will **not** choose the edge connecting B and G, because G is already in the network and building it would only be extra cost!

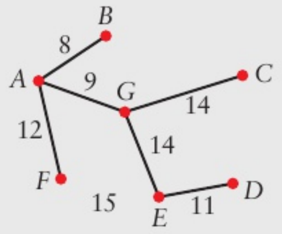
4 From the connected vertices we now have, the next closest is E from G.

5 The next closest is D from G.

6 The next closest is C from G.



And finally:



You have now found the minimum spanning tree (MST).

The **weight** of the tree is found by adding the values of the different edges:

$8 + 9 + 12 + 14 + 11 + 14 = 68$

This means that the total cost of connecting these farmhouses to the electricity network is 68 thousand dollars.

The approach to find the MST requires a bit of discipline, but it should make complete sense. For a check, compare the total cost found here with the costs of the three trees suggested at the beginning: 75, 76, and 83. Clearly, the tree we found has less cost than any of the others.

Hint

In Step 4, both C and E have the same weight from G. It does not matter which order we add them in. The end result will be the same. Try adding C first and see if you get the same result.

Fact

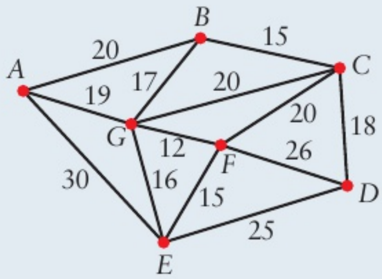
The method applied in the worked example is called an algorithm, which is derived from the same word as algebra: al-Khwarizmi. This is the (transliterated) name of a famous 9th century mathematician from Persia. Algorithms are the processes and procedures that are at the heart of computer programs.

Reflect

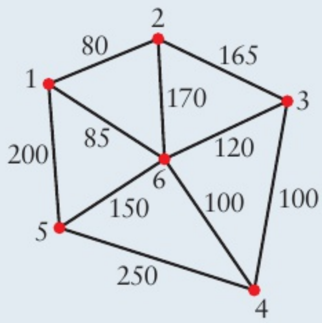
Can you summarise the approach explained in this example in your own words? Verify that if you start at a different vertex, you will obtain the same MST.

Practice questions 11.4

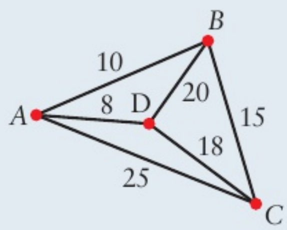
1 Find the minimum spanning tree of this graph. Start at vertex A and use the method described in the previous example.



2 The network shows the distances, in metres, between a number of cabins. Find the shortest route to walk to all of the cabins



3 In this graph, vertex D represents a distribution centre of biogas. Vertices A, B and C are to receive biogas from D through edges. The weights of the pipelines corresponds to the cost of the pipelines in thousands of euros.



   a Find the cost of connecting A, B and C directly to D.

   b Find the cheapest solution to connect all vertices.

   c How much money is saved using this solution?

   d What is the mathematical name of this graph?

4 The diagram shows a map of the United Arab Emirates. The weights on the edges are the distances in kilometres between the given cities.

Fact

This graph is also known as a wheel graph – can you see why? Its mathematical name is W_6.

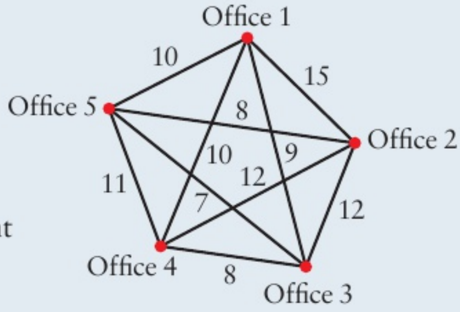
a Why is it unlikely that the edges represent actual roads?

b What is the shortest distance to travel around all five cities?

c For your answer to part b, does it make a difference where you start?

d Estimate the distance from Abu Dhabi to Al Fujayrah. How did you arrive at this answer?

5 A company has five offices in a town. The manager of the company wants to connect the offices by glass fibre to optimise the common computer network. The weights of the edges represent the cost of the connections in thousands of pounds.

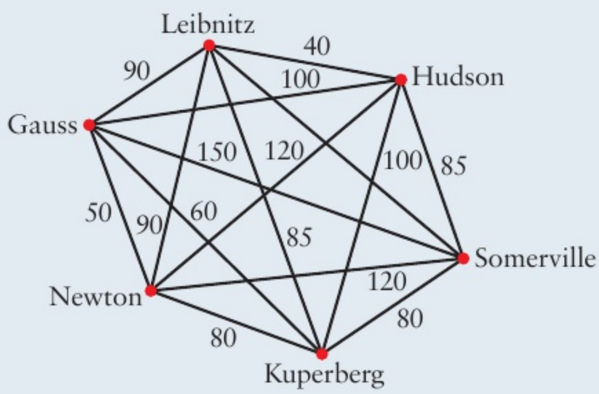


a What is the cheapest solution for this company?

b The best solution is not always the cheapest. What other considerations could be important here?

c What is the mathematical name of this graph?

6 A network of bus routes has been put in place to connect the towns in the diagram. The weights of the edges represent the travel times between towns in minutes.

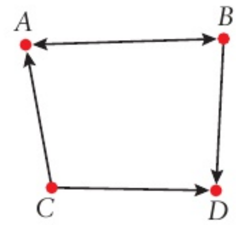


a Find the minimum spanning tree of this graph.

b In what town would you put the main bus station? Explain your reasoning.

c Due to economic cuts, the trajectories of the MST are the only ones offered. How does this affect the passengers travelling from Newton to Hudson?

11.5 Directed graphs and problem solving



A directed graph has arrows on the edges showing the direction of travel. An example of a directed graph is shown here. In a directed graph, the edges are called **arcs.**

As you can imagine, this changes the problem of finding an Euler circuit. In this section, we are going to explore exactly how the limitations on the direction of travel affect the problems we have looked at earlier in this chapter.

Explore 11.8

Is there an Euler circuit or trail in the graph above? What is the shortest path to travel to each vertex, starting from A? Is it possible to go around each vertex without visiting any vertex twice? If you could add an edge to make it possible, where would you put this edge, and with what direction(s)?

The most practical applications of directed graphs are in a traffic situation, such as a road map of a city, or as a **flowchart** to model an event, such as in the worked example below, which models the making of a cup of instant coffee.

Worked example 11.5

Make a flowchart of the process of making a cup of instant coffee.

Solution

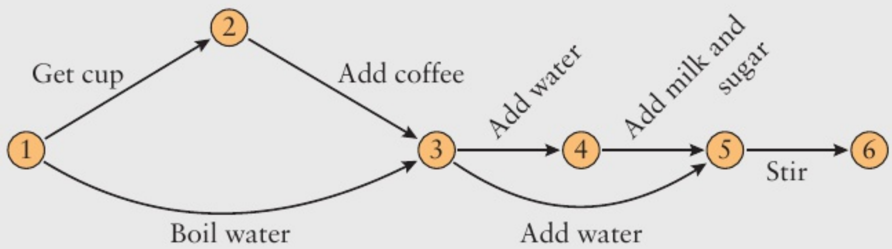
We need to make a flowchart of this process in time, which consists of a sequence of different steps, some of which follow other steps.

Start by identifying the different actions involved in the process and indicate which steps can be parallel and which need to wait for other steps to be completed.

We will record the different actions of the process in a table and suggest the order of the steps:

Action	Step number	Condition
Get cup	1	Start – parallel with boiling water
Boil water	1	Start
Add coffee	2	Before adding water, after getting cup
Add water	3	After boiling water and adding coffee
Add milk and sugar	4	Optional step after adding water
Stir	5	At the end

Now these steps can be translated into a flowchart, in which time flows from left to right:



Looking back, could we change anything about this diagram? Why can't we change the direction of the arrows? What can you say about arrows that are running parallel to other arrows?

As we can see, directed graphs can be used to organise processes in time where one step depends on another step. In the diagram, we can see how the order of operations is important to obtain the desired result. In this example, this is quite straightforward, but when operations become more complicated, it is quite helpful to make such a flowchart.

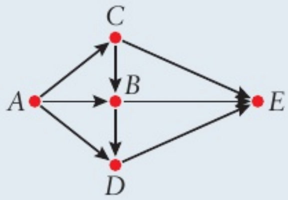
Reflect

What do you think the strengths and weaknesses of flowcharts are? Think of practical applications.

Practice questions 11.5

1 Make a flowchart of your own morning routine, from waking up to leaving home to go to school.

2 In the given graph:



a list the vertices

b list the arcs

c list the paths you can take to get from B to E

d list the paths you can take to get from A to E.

3 Answer the following questions for this graph.

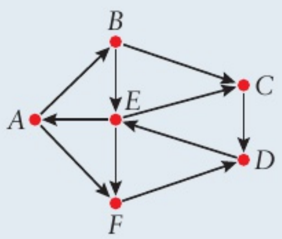
a How many vertices are there?

b How many arcs are there?

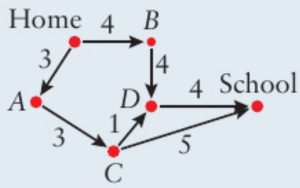
c List the different paths from:

i A to C

ii A to D



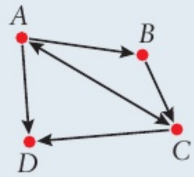
4 This graph shows the different roads Kim can cycle to school. The weights represent the time it takes between two vertices in minutes.



a What do the vertices represent?

b List the quickest path(s) for Kim to get to school. What is the total time required for the journey?

5 Answer the following questions for this graph.

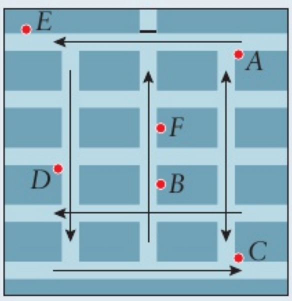


a Verify that it is not possible to make a circuit visiting each vertex.

b What could you change to make this possible?

c Is it possible to go around each arc once and return to where you started?

6 The diagram shows the street plan for a city centre. The arrows show the direction of traffic. The red dots represent tourist highlights. You are a tour guide taking a group around the city centre in a bus.



a Where must the tour end? Why?

b Which route are you going to take to visit the different highlights, starting at A?

c Are different routes possible?

7 List the steps involved in making your own pizza using a ready made pizza base. Put the steps in a flowchart.

8 You have received funding to plant trees in the school's garden. List the steps involved in making this happen and put them in a flowchart.

9 Your school has discussed the importance of not using single-use plastics. How could you go about this? List the steps and put them in a flowchart.

For more information visit:

10 Your school has discussed the importance of reducing the consumption of meat. How could you go about this? List the steps and put them in a flowchart.

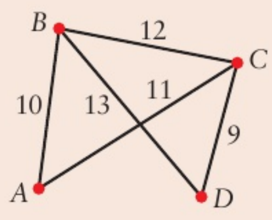
For more information visit:

Self assessment

- I understand the definition of a vertex, edge and degree of a vertex.
- I understand what a subgraph is.
- I understand what a tree and spanning tree are.
- I know how many edges a tree with n vertices has.
- I know what a complete graph is.
- I know how to find the number of edges in a complete graph.
- I know what an Euler trail is and what conditions a graph must fulfil to have one.
- I know what an Euler circuit is and what conditions a graph must fulfil to have one.
- I can solve problems involving directed graphs.
- I know how to find the minimum spanning tree of a weighted graph.

Check your knowledge questions

1 Looking at the graph, answer the following questions.



a How many vertices are there?

b How many edges are there?

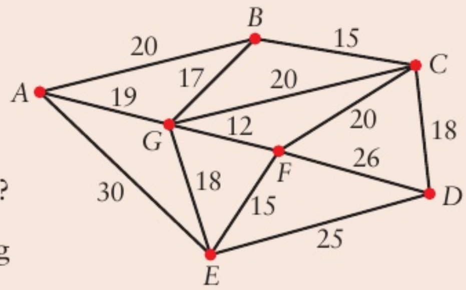
c What are the degrees of the vertices?

d Is an Euler trail possible? Why, or why not?

e Compare the sum of the degrees to the number of edges. What do you notice?

2 Using the same graph as in question 1, draw the minimum spanning tree. What is its weight?

3 Looking at the graph, answer the following questions.

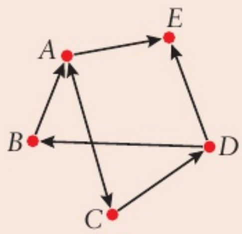


a What kind of graph is this?

b What is the degree of vertex G?

c What is the minimum spanning tree and what is its weight?

4 Looking at the graph, answer the following questions.



a What kind of graph is this?

b Suggest a way to travel from:

i A to D

ii B to D

c Verify that it is not possible to visit each vertex and return to where you started.

d Is it possible to travel along each edge and end where you started? If not, suggest which edge(s) you should add to make this possible (with direction).

5 a How many edges are there in K_7?

b Draw K_7 and check that your answer to part a was correct.

6 How many edges does a tree with n vertices have?

Chapter 1 answers

Do you recall?

1 Integers: 3, −5
Irrational: $\sqrt{2}$
Natural no: 3
Rational no: 3.2, 2.333…, $\frac{3}{5}$, 2^{-3}
Real no: 3, −5, 3.2, 2.333…, $\sqrt{2}$, $\frac{3}{5}$, 2^{-3}

2 24 is divisible by 3 but not by 5

3 The result is 9. Divide and multiply have the same priority according to BODMAS so we calculate from left to right.

Practice questions 1.1

1 Used (Y) for yes, (N) for no

Number	Natural	Integer	Rational	Irrational	Real
101	Y	Y	Y	N	Y
−6	N	Y	Y	N	Y
π^2	N	N	N	Y	Y
2.13	N	N	Y	N	Y
$\frac{19}{3}$	N	N	Y	N	Y
-2^3	N	Y	Y	N	Y
$\sqrt{6}$	N	N	N	Y	Y
-5^2	N	Y	Y	N	Y
$(-1.1)^2$	N	N	Y	N	Y
0.111…	N	N	Y	N	Y
$3.\dot{5}$	N	N	Y	N	Y
$\sqrt{64}$	Y	Y	Y	N	Y

2 a 29, 31, 37, 41, 43, 47, 53, 59 and 61. Total 9 prime numbers
b 9, 16, 25, 36, 49. Total 6 square numbers
c Only 1, 64. There are two.
d There are 19 numbers: 9, 19, 29, 39, 49, 59, 69, 79, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99 (there are 2–9 digits in 99!)

3 a HCF (12, 30) = 6, LCM (12, 30) = 60
b $540 = 2^2.3^3.5$
c $6 = 2 \times 3$, therefore 4×6 is divisible by both 2 and 3. So $4 + x + 6 = 10 + x$ must be divisible by 3. Thus, possible x values are 2, 5, 8
d 2341 is a prime number since 1 and itself are the only two factors.

4 a $\dfrac{\frac{1}{2}+\frac{2}{3}}{\frac{1}{2}-\frac{2}{3}} = \dfrac{\frac{7}{6}}{\frac{-1}{6}} = -7$
b $\frac{3}{4} \times 1\frac{4}{5} = \frac{3}{4} \times \frac{9}{5} = \frac{27}{20}$
c $1\frac{12}{13} \div \frac{5}{26} = \frac{25}{13} \div \frac{5}{26} = \frac{25}{13} \times \frac{26}{5} = \frac{650}{65} = 10$
d $\left(\frac{3}{5}\right)^2 \div \frac{1}{2} \times \frac{5}{2} = \frac{9}{25} \times \frac{2}{1} \times \frac{5}{2} = \frac{90}{50} = \frac{9}{5}$ or 1.8

5 a $2.333\ldots = \frac{21}{9}$ b $3.222\ldots = \frac{29}{9}$
c $11.2333\ldots = \frac{1011}{90}$ d $5.121212\ldots = \frac{507}{99}$

6 a $121.2 + 1.212 - 12.12 = 110.292$
b $2 \times 30 \times 12 = 720$
c $(2 \times 3)^2 = 36$
d $\sqrt{1.44} = 1.2$

7 a $\{2^3 \div -2\} \times \{(-2) \times -3^2\}$
$= \{8 \div -2\} \times \{(-2) \times -9\}$
$= -4 \times 18 = -72$
b $(16 - 3^3 \times 2 \div (-6)) - 25$
$= (16 - 27 \times 2 \div (-6)) - 25$
$= (16 - 54 \div (-6)) - 25 = (16 - (-9)) - 25$
$= 25 - 25 = 0$
c $\sqrt{81} \div 3^2 - 3 = 9 \div 9 - 3 = 1 - 3 = -2$
d $\dfrac{\frac{-5}{4} + 1\frac{2}{3}}{2} \div \frac{11}{24} = \dfrac{\frac{-5}{4} + \frac{5}{3}}{2} \div \frac{11}{24} = \dfrac{\frac{5}{12}}{2} \div \frac{11}{24}$
$= \frac{5}{24} \div \frac{11}{24} = \frac{5}{24} \times \frac{24}{11} = \frac{5}{11}$

Practice questions 1.2

1 a $y = 3x + 2$

x	−2	0	2	4
y	−4	2	8	14

b $w = 5z - 4$

z	−1	1	3	5
w	−9	1	11	21

2 a $y = 2x + 1$ b $y = 3x + 2$

3 a $10x$ b $6y$ c $5a + 2b$

d $-21x$ e $4w$ f $9a$

4 a $\frac{7x}{6}$ b $\frac{7x}{12}$ c $\frac{8x^2}{15}$

d $\frac{18}{10} = \frac{9}{5}$ or 1.8

5 a $P = 14x + 2$ $A = 12x^2 + x - 6$

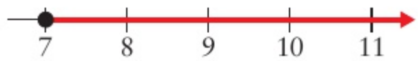
b $P = 12x$ $A = 4\sqrt{3}x^2$

6 a $x = 11$ b $x = 23$ c $x = 4$

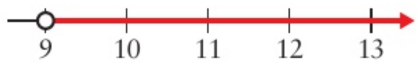
d $x = 12$

7 a $x \leq 4$

b $x \geq 7$



c $x > 9$



d $x < 6$

8 a $12\,x = 96$, $x = 8$

b $5y + 3 = 38$, $y = 7$

c $x + x + 1 + x + 2 = 96$, $x = 31$

d $m + 3 + m = 27$, $m = 12$ (Mette), $m + 3 = 15$ (Eric)

Practice questions 1.3

1 a $x° + 75° + 135° + 90° = 360°$, $x° = 60°$

b $82° + y° + 34° = 180°$, $y° = 64°$

c $31° + 55° + z° = 90°$, $z° = 4°$

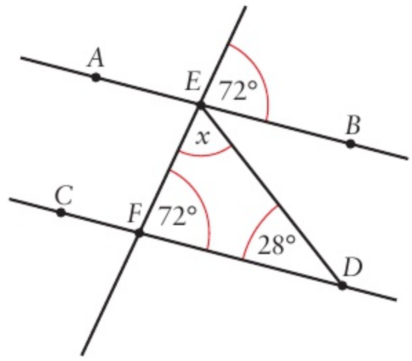
d $b° = 115°$ (vertically opposite angles), $a° + 115° = 180°$ (straight line angles), $a° = 65° = c°$ (vertically opposite angles)

2 a $x° = 130°$ (corresponding angles), $y° + 130° = 180°$ (supplementary angles), $y° = 50°$

b $x° = 100°$ (vertically opposite angles) $x° = y° = 100°$ (alternate angles)

c $x° = 65°$ (alternate angles), $x° + y° = 180°$(supplementary angles), $65° + y° = 180°$, $y° = 115°$

d



$x° + 72° + 28° = 180°$, $x° = 80°$

3 a $72° + x° + x° = 180°$, $2x° = 108°$, $x° = 54°$

b $23° + 33° + y° = 180°$, $y° = 124°$

c $90° + x° + x° + 2x° = 360°$, $4x° = 270°$, $x° = 67.5°$

d $47° + 68° + 25° + x° = 360°$, $x° = 220°$

4 a $P = 3 \times 4 = 12$ units

b $EG^2 = 3^2 + 6^2 = 45$
$EG = \sqrt{45} = 3\sqrt{5} = 6.7$
Thus $P = 3 + 6 + 6.7 = 15.7$ units

c $C = 2 \times \pi \times 2 = 4\pi$ units

d $P = 5 + 11 + 13 = 29$ cm

5 a $A = \pi \times 3^2 = 9\pi$ unit^2

b $A = 4 \times 3 + \frac{1}{2} \times 1 \times 3$
$A = 13.5$ unit^2

c $A = \frac{1}{2} \times 2 \times 10 = 10$ units^2

d $A = 4 \times 3 = 12$ units^2

6 Surface area $= 2.\ (2 \times 3 + 3 \times 3 + 2 \times 3)$ $= 2.\ 21 = 42$ cm^2

Practice questions 1.4

1 a $S = \{$head, tail$\}$ b $S = \{1,2,3,4,5,6\}$

c $S = \{$March, May$\}$ d $S = \{$Wednesday$\}$

2 a $P(a) = \frac{2}{7}$ b $P(\text{vowel}) = \frac{3}{7}$

c $P(k) = \frac{0}{7} = 0$ d $P(\text{not b}) = 1 - \frac{1}{7} = \frac{6}{7}$

3 a $100\% - 40\% = 60\%$

b $\frac{5}{6}$ c $\frac{1}{2}$ d $1 - 0.8 = 0.2$

4 a $S = \{TTT, TTH, THT, HTT, HHT, HTH, THH, HHH\}$

b $P(TT) = \frac{3}{8}$

c $P(TTT) = \frac{1}{8}$

d P(at least one tail) = 1 − P(no tail)
$= 1 - \frac{1}{8} = \frac{7}{8}$

5 a

Outcome	Tally	Frequency
0	1111	4
1	111	3
2	1111	4
3	1	1
4	1	1
5	111	3
6	1	1
8	1	1
	Total	18

b Sum of the data is 47

c Mode is 0 and 2

d Mean is $\frac{47}{18} = 2.6111...$

e Range = 8 − 0 = 8

f Median is the average of 9th and 10th numbers, that is $\frac{2+2}{2}$

6 a i 29 ii 29.5 iii 33 − 20 = 13

b 20

Check your knowledge questions

1 a Multiples of 11: 11, 22, 33, 44, 55, 66, 77, 88, 99, 110, 121, 132

b Factors of 24: 1, 2, 3, 4, 6, 8, 12, 24

c HCF (45, 60) = 15

d LCM (11, 12) = 132

2 a 11, 13, 17, 19, 23, 29, 31, 37

b 4, 6, 8, 9, 10, 12, 14

c $240 = 2^4.3.5$

3

Number	Whole	Integer	Rational	Irrational	Real
79	Y	Y	Y	N	Y
−11	N	Y	Y	N	Y
3π	N	N	N	Y	Y
3.5	N	N	Y	N	Y
$\frac{-1}{5}$	N	N	Y	N	Y
$(-2)^4$	Y	Y	Y	N	Y
$\sqrt{16}$	Y	Y	Y	N	Y
$\sqrt{15}$	N	N	N	Y	Y
$(2.3)^2$	N	N	Y	N	Y
3.222…	N	N	Y	N	Y

4 a $-8 - 4 \div 2 \times 3 = -8 - 2 \times 3$
$= -8 - 6 = -14$

b $\sqrt{25} - 3^3 \div 3 + 4 = 5 - 27 \div 3 + 4$
$= 5 - 9 + 4 = -4 + 4 = 0$

c $(3 \times 5 - 2^2) - 13 + \frac{4}{2} = (15 - 4) - 13 + 2$
$= 11 - 13 + 2$
$= -2 + 2 = 0$

5 $12.1212... = \frac{1200}{99} = \frac{400}{33}$

6 For $y = 3x + 7$

y	0	−1	1	2
x	7	4	10	13

7 $y = 5x - 5$

8 a $2x - 3y + x - 11y = 3x - 14y$

b $\frac{x}{2} - \frac{x}{3} = \frac{3x}{6} - \frac{2x}{6} = \frac{x}{6}$

c $16ab^2 \div 8a^2b = \frac{2b}{a}$

d $\frac{-25x \times 6y}{10xy} = \frac{-150xy}{10xy} = -15$

9 a $\frac{x}{3} - 5 = 11, \frac{x}{3} = 16, x = 48$

b $2(x - 1) + 3(1 - x) = -1$
$2x - 2 + 3 - 3x = -1$
$-x + 1 = -1, x = 2$

c $5(x + 3) = 2(3x - 1)$

$5x + 15 = 6x - 2$

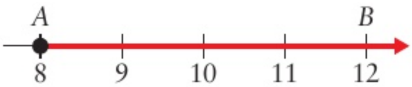
$17 = x$

d $\dfrac{4x + 5}{3} = 3$

$4x + 5 = 9$

$4x = 4,\ x = 1$

10 a $2x - 5 \geq 11,\ 2x \geq 16,\ x \geq 8$



b $\dfrac{y}{2} - 7 \leq 3,\ \dfrac{y}{2} \leq 10,\ y \leq 20$

c $2(z - 3) < z + 1,\ 2z - 6 < z + 1,\ z < 7$

d $\dfrac{w}{3} + \dfrac{w}{2} > 5,\ \dfrac{5w}{6} > 5,\ 5w > 30,\ w > 6$

11 a $x + 47 + 83 + 160 = 360$

$x + 290 = 360,\ x = 70$

b $z° + 105° = 180°$(straight angles), $z° = 75°$, $z = 75$

$x° = 105°$ (vertically opposite angles), $x = 105$

$y° = 105°$ (corresponding angles or alternate angles), $y = 105$

c $x° + 2x° + 3x° = 180°,\ 6x° = 180°,$ $x° = 30°,\ 2x° = 60°,\ 3x° = 90°;\ x = 30,$ $2x = 60,\ 3x = 90$

d Regular hexagon has $(6 - 2) \times 180° = 720°$ as sum of interior angles.

Thus $6x° = 720°,\ x° = 120°,\ x = 120$

12 $2\pi r = 2\pi r^2$

$\dfrac{2\pi r}{2\pi r} = \dfrac{2\pi r^2}{2\pi r}$

$1 = r$

13 a $P(\text{red}) = \dfrac{4}{12} = \dfrac{1}{3}$ **b** $P(\text{black}) = \dfrac{3}{12} = \dfrac{1}{4}$

c $P(\text{not blue}) = \dfrac{7}{12}$ **d** $P(\text{yellow}) = \dfrac{0}{12} = 0$

14 a There are 6 outcomes, tally and frequency column show how many times each outcome repeats.

Outcome	Tally	Frequency
11	111	3
12	~~1111~~ 1	6
13	111	3
14	~~1111~~	5
15	1111	4
16	1111	4
	Total	25

b Range = 16 − 11 = 5

Mean = (3.11 + 6.12 + 3.13 + 5.14 + 4.15 + 4.16) ÷ 25 = 13.52

Mode = 12

Median is the middle number which is the 13th piece of data "14"

Chapter 2 answers

Do you recall?

1 a You can describe this as a ratio of percentages. For example 40% : 60% for boys to girls means that for every 2 boys there are 3 girls.

500 g to 2 kg is $\dfrac{500}{2000} \times 100 = 25\%$

2 $\dfrac{12}{15} = \dfrac{4}{5},\ \dfrac{56}{70} = \dfrac{8}{10} = \dfrac{4}{5}$

This is not equal to 75% which is $\dfrac{3}{4}$

Practice questions 2.1

1 a $\dfrac{3}{4} = \dfrac{75}{100} = 75\%$

b $\dfrac{12}{20} = \dfrac{60}{100} = 60\%$

c $\dfrac{125}{100} = 125\%$

d $1\dfrac{11}{25} = \dfrac{36}{25} = \dfrac{144}{100} = 144\%$

2 a $34\% = \frac{34}{100} = \frac{17}{50}$

b $15.4\% = \frac{15.4}{100} = \frac{154}{1000} = \frac{77}{500}$

c $111\% = \frac{111}{100}$

d $5\% = \frac{5}{100} = \frac{1}{20}$

3 a $0.23 = \frac{23}{100} = 23\%$

b $12.5 = \frac{1250}{100} = 1250\%$

c $0.5 = \frac{50}{100} = 50\%$

d $0.01 = \frac{1}{100} = 1\%$

4 a $\frac{5}{100} \times 1200 = 60\,\text{kg}$ b $\frac{25}{100} \times 100 = 25\,\text{m}$

c $\frac{15}{100} \times 60 = 9$ min d $\frac{1.1}{100} \times 11 = 0.121$

5 a $25\% + 100\% = 125\%$

$25 \times \frac{125}{100} = 31.25$ euro

b $30\% + 100\% = 130\%$

$150 \times \frac{130}{100} = 195\,\text{m}$

c $100\% - 40\% = 60\%$

$2000 \times \frac{60}{100} = 1200\,\text{mm}$

d $100\% - 5\% = 95\%$

$\frac{95}{100} \times 40 = 38\,\text{h}$

6 $\frac{805 - 544}{805} \times 100\% = 32.4\%$

7 $6500 \times (100\% - 9\%) = 5915$ vehicles

8 $\frac{8}{100} \times 2\,000\,000 = 160\,000$ euro

9 Total solar power capacity is 627 000

a percentage solar power for China is
$\frac{204\,700}{627\,000} \times 100 = 32.647 \approx 32.6\%$

b percentage solar power for Germany is
$\frac{49\,200}{627\,000} \times 100 = 7.846 \approx 7.85\%$

c percentage solar power for all other countries is $\frac{121\,600}{627\,000} \times 100 = 19.393$
$\approx 19.4\%$

d percentage solar power for Australia is
$\frac{146\,00}{627\,000} \times 100 = 2.328 \approx 2.33\%$

Practice questions 2.2.1

1 a $2:3$ b $3:2$ c $2:5$ d $3:5$

2 a $5:4$ b $4:5$ c $5:9$ d $4:9$

3 Tom gets €108, Clara gets €162.
The difference is €54.

4 a $100:25$ or $4:1$ b $300:100$ or $3:1$

c $1000:100$ or $10:1$ d $500:25$ or $20:1$

5 a $228:304$ or $57:76$

b $2808:5452$ or $702:1363$

c $\frac{57}{76} > \frac{702}{1363}$

6 a $A:B = 88:53$ b $B:C = 53:6$

c $AB:AC = 141:147$ d $88:59$

7 Aiswarya is correct since the angles of the triangle are $40°:60°:80° = 2:3:4$

8 a $3:5$ b $1:7$ c $8:17$ d $3:5$

9 a $NL:ML = 50:34 = 25:17$

b $NL:BL = 50:25 = 2:1$

c $BL:ML = 25:34$

d $BL:NL = 1:2$

Practice questions 2.2.2

1 a $x = 30$ b $y = 45$ c $w = 22$ d $z = 27$

2 a $5:1$ b $2.2:1$ c $2.\overline{6}:1$ d $8:1$

3 a $x = 173$ b $y = 24$

c $z = -6$ d $w = 1.11\ldots$

4 a 7 038 000 000 people

b 782 000 000 people

5 $420 \div (3 + 4) = 60$, $\frac{3 \times 60}{4 \times 60} = \frac{180}{240}$, thus Ayubke receives €180, Olga receives €240.

6 a 900 cm or 9 m b 360 cm or 3.6 m
c 600 m or 0.6 km d 75 000 mm or 75 m

7 a No b No c Yes d No

8 New length 16 m, new width 20 m

Practice questions 2.2.3

1 a 125 km/h b 25 kg/m^2
c 125 L/min d 12.7 kr/L

2 a 126 000 tonnes b 1 095 000 tonnes

3 20 km

4 13 hours and 20 minutes

5 40 weeks

6 100 m is 10.44 m−s and 200 m is 10.42 m−s which is 0.02 m−s better.

7 They are almost the same since Car A uses 12 km−l and Car B uses 12.08 km−l. But Car B is more economical.

8 a 129 600 000 km
b 11 111 111 111 hours or 462 962 963 days

Check your knowledge questions

1 a 5 : 3 b 6 : 17 c 7 : 11 d 33 : 50

2 a $x = 10.5$ b $y = 24$
c $z = 1$ d $w = 17$

3 a 24 dogs b 40%

4 a 200 983 : 980 080 b 153 997 : 200 983
c 72 133 : 980 080 d 200 983 : 1 407 193

5 a 20 km−h b 2 kg−m^2
c 3 g−l d 3 : 4

6 a 0.04 m or 4 cm
b $0.01\overline{6}$ km or $16.\overline{6}$ m
c $0.2\overline{3}$ m or $23.\overline{3}$ cm
d 10 cm

7 $x = 15$ and $y = 90$

8 Thomas is right because the angles are 45°, 45°, 90° which are angles of an isosceles, right-angled triangle in the ratio of 1 : 1 : 2.

9 a No b Yes, it is $\frac{1}{3}$
c No d No

10 a Route 1: 103.1 km/h, route 2: 97.8 km/h
b Route 2 is better for the petrol consumption, and time.

11 a 236 402 : 25 808 or 118 201 : 12 904
b 13 413 : 37 506 or 4471: 12 502
c 61 357 : 105 466
d 10 512 : 15 452

12 a 0.006 m or 6 mm
b 0.012 m or 12 mm
c 0.12 cm or 1.2 mm
d 2.5 mm

Chapter 3 answers

Do you recall?

1 a 1 b 1 c −7

2 a 12 b −4 c 36

3 a 12 b 14 c 6

4 a $2^4 \times 3$ b $2^2 \times 3 \times 5$ c 2×7^2
d $3^2 \times 5 \times 7$

Practice questions 3.1.2

1 a $4x + 2y + 3$ b $6p + 8q$
c $3a + 10b + 5c + 2$ d $9x + 11$
e $6x + 4$ f $x + 5$
g $x + 3$ h $-5a - 6b - c$
i $5x^2 - 9x + 9$ j $4x + 4$
k $x - 5$ l $8a + 2b - 5c + 5$

2 a $2x + 2y$ b $6x + 6$ c $4x + y + 17$

Practice questions 3.1.3

1 a $5z$ b x^2 c y^3 d z^5
e $21a$ f $24b^2$ g xyz h $24abc$
i $63x^{13}$ j $6p^5q^6r^3$

2 a $7x$ b xy c $10x^2$ d $24x^2y$

Practice questions 3.1.4

1 a p^4 b b^4 c $3t^6$

d $\frac{1}{5}z^7$ or $\frac{z^7}{5}$ e $4k^8$ f $\frac{2}{3x^3}$

g $\frac{7}{6}a^4b^7$ h $\frac{8q^6}{p^3}$

2 a $4x^3$ b $9y^3$ c $3z^2$

d $\frac{9}{5}a^3b^7$ e $\frac{4p}{3q^9}$ f m^3p^5

3 a $\frac{4x^6y^4}{35z^5}$ b $\frac{14x^2z^8}{15y^3}$

c $\frac{128a^7}{135b^7c^3}$ d $143x^4y^{10}z$

4 a $3.5x$ b $3x^5y^4$ c $1.5x^7y$

Practice questions 3.2.1

1 a $6x + 24$ b $2 - 3x$

c $4x + 6$ d $12x - 15$

2 a $21x + 15$ b $-38x - 1$

c $19x - 51$ d $-23x - 1$

3 a 13 b $15x - 14$ c $52x - 38$

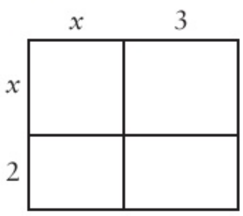
4 a $20a + 5b$ b $56c - 16d$ c $6e - 4.5$

5 a $50 - 2x$ cm b $60 - 2x$ cm

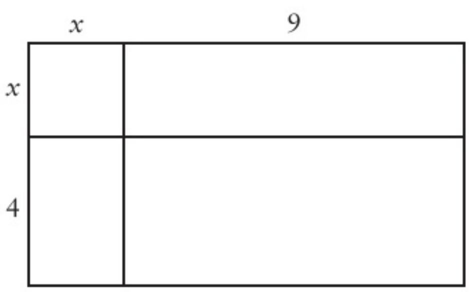
c $50x - 2x^2$ cm^2 and $60x - 2x^2$ cm^2

Practice questions 3.2.2

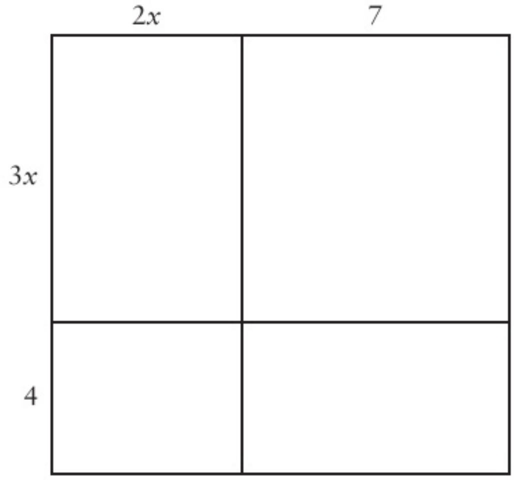
1 a



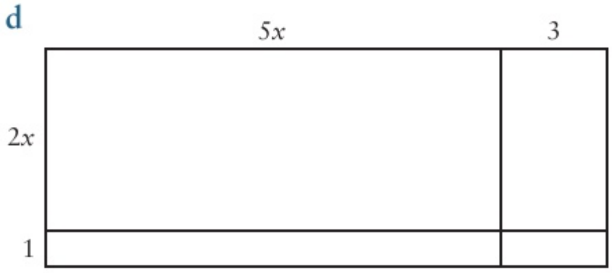
b



c



d



2 a $x^2 + 5x + 6$ b $x^2 + 13x + 36$

c $x^2 + 7x + 6$ d $x^2 + 5x - 24$

e $x^2 - 5x - 14$ f $x^2 - 2x - 63$

g $x^2 + 9x - 36$ h $x^2 - 10x + 24$

i $x^2 - 16x + 55$ j $x^2 + 10x + 25$

k $x^2 + 2x + 1$ l $x^2 - 12x + 36$

m $x^2 - 8x + 16$ n $x^2 - 64$

o $x^2 - 4$ p $x^2 - 49$

3 a $5x^2 + 17x + 6$ b $4x^2 + 7x - 2$

c $14x^2 + 36x - 18$ d $4x^2 - 9$

e $12x^2 - 120x + 252$ f $-18x^2 + 15x + 12$

g $2x^3 + 12x^2 + 5x + 30$ h $x^6 + 7x^3 + 10$

4 a $x^3 + 6x^2 + 11x + 6$

b $x^3 + 8x^2 + 22x + 21$

c $24x^3 - 42x^2 - 45x$

d $8x^3 - 22x^2 + 17x - 3$

e $x^2 - 5x$

f $12x^2 - 6x$

5 a $x^3 + 2x^2 - 15x$ b $x^3 - 13x^2 + 40x$

c $x^3 + 17x^2 + 80x + 100$

6 a $100 - x$ b $5x - 0.05x^2$

7 a $x^2 - 7x\,\text{cm}^2$

b $1000x^3 - 4000x^2 - 21\,000x\,\text{cm}^3$

8 a $4x^2 - 220x + 3000$

b $4x^3 - 220x^2 + 3000x$

c $3000 - 4(x^2)$

9 a $6y^2$ b $x - y$

c $225 - x + \frac{1}{2}y$ d $225x - x^2 + \frac{1}{2}xy$

e $\frac{225}{4}x - \frac{1}{4}x^2 + \frac{1}{8}xy$

Practice questions 3.3

1 $(2x + 4)(x + 4)$ or $(x + 2)(2x + 8)$

2 a HCF $= b$; $b(ac + ad + cd)$

b HCF $= 6$; $6(2x + 3)$

c HCF $= 10x$; $10x(3x - 2)$

d HCF $= 4x^7y^3$; $4x^7y^3(4y^7 + 7x^2)$

e HCF $= 2x^3$; $2x^3(2x^3 - 6x^2 + 3)$

3 a $(x + 1)(x + 7)$ b $(x + 2)(x + 4)$

c $(x + 3)(x + 5)$ d $(x + 5)(x + 12)$

e $(x - 2)(x - 1)$ f $(x - 5)(x - 3)$

g $(x - 14)(x - 3)$

h $(x - 4)(x - 4) = (x - 4)^2$

4 a $(x - 10)(x + 4)$ b $(x - 1)(x + 12)$

c $(x - 8)(x + 2)$ d $(x - 9)(x + 2)$

e $(x - 4)(x + 6)$ f $(x + 9)(x - 8)$

g $(x - 3)(x + 3)$ h $(x - 6)(x + 6)$

5 a 3; $3(x - 5)(x - 1)$ b 4; $4(x + 1)(x + 6)$

c 2; $2(x + 3)(x + 5)$

6 a $x - 1, x + 1$

b $(x + 1)^2 - (x - 1)^2$
$= x^2 + 2x + 1 - (x^2 - 2x + 1) = 4x$
$= 4 \times 1\,000\,000 = 4\,000\,000$

c Let $x = 2\,000\,000$; $\frac{(x + 2)^2 - (x - 2)^2}{(x - 2)(x)(x + 2)}$
$= \frac{8x}{x(x - 2)(x + 2)} = \frac{8}{x^2 - 4}$
$= \frac{8}{3\,999\,999\,999\,996} = \frac{2}{999\,999\,999\,999}$

7 a 23×37 b 67×73 c 71×89

Check your knowledge questions

1 $a = 6, b = 10$

2 a 9; 3; 9

b $\frac{1}{2}$; 3; $8x$; $\frac{1}{2}x$; $8x$; 3

c 3; 1; $3x \times x + 3x \times 1 - 2 \times x - 2 \times 1$; (-2)

3

	$3x$	2
$1x$		
4		

$3x^2 + 14x + 8$

4 a i $P = 16x - 14\,\text{cm}$; $A = 15x^2 - 35x\,\text{cm}^2$

ii $P = 160x - 140\,\text{mm}$;
$A = 1500x^2 - 3500x\,\text{mm}^2$

b i $P = 26x - 10\,\text{cm}$;
$A = 36x^2 - 5x - 24\,\text{cm}^2$

ii $P = 260x - 100\,\text{mm}$;
$A = 3600x^2 - 500x - 2400\,\text{mm}^2$

5 a p b p, q

c p, q, r (assuming q is not negative)

6 a $100A^2 + 20AB + B^2$

b $100A^2 + 20A \times 5 + 5^2$
$= 100A^2 + 100A + 25 = 100A(A + 1) + 25$

c 225, 625, 1225, 2025, 3025, 4225, 5625, 7225, 9025

7 a $5x^2 - 2x\,\text{km}$ b $\frac{5x^2 - 2x}{5x - 4}\,\text{h}$

c $\frac{300x^2 - 120x}{5x - 4}$ min for the slower journey.
Faster journey time is $60\,x$ min.
Difference is $\frac{120x}{5x - 4}$ min

8 a $48x$ b $96x^2$ c $64x^3$

d $48x + 12$; $96x^2 + 48x + 6$;
$64x^3 + 48x^2 + 12x + 1$

9 a i $6x - 9$ ii $23x - 16$; $28x - 5$
b $28x^3 - 68x^2 + 15x + 36$;
$98x^3 - 35x^2 - 136x - 48$

10 a $(x + 2)(x + 5)$ b $(x - 12)(x + 7)$
c $(x - 6)(x + 9)$ d $(x - 9)(x - 4)$
e $(3 - x)(1 + x)$ or $-(x - 3)(x + 1)$
f $(x - 7)^2$ g $(x - 11)(x + 11)$
h $(x + 6)^2$ i $(3x - 2)(3x + 2)$
j $9(x - 2)(x + 2)$ k $3(x - 3)(x - 2)$
l $4(x + 2)(x + 6)$

11 $4x + 4$ by $x + 3$; $4x + 12$ by $x + 1$;
4 by $x^2 + 4x + 3$

12 a 113×127 b $3^2 \times 23 \times 193$

Chapter 4 answers

Do you recall?

1 64, 1600, 1.44, 1225

2 7, 11, 50, 1.2

3 74.3 cm²

4 Angles 2 and 4 are alternate angles.
Angles 3 and 5 are corresponding angles.
Angles 1 and 3 are vertical (or vertically opposite).
Angles 3 and 4 are co-interior.
Angles 1 and 2, 2 and 3, and 4 and 5 are supplementary.

5 $x^2 + 6x + 9$, $x^2 + 2xy + y^2$, $x^2 - 2xy + y^2$,
$9x^2 + 30xy + 25y^2$

Practice questions 4.1

1 *ABC*, *QPR*, *SUT* (right angle at middle vertex)

2 *AC*, *QR*, *ST*

3 a Student's table
b last two columns should be equal

4 a $A_1 + A_2 = A_3$ b $a^2 + b^2 = c^2$

5 a ΔAB_2C b 90°

Practice questions 4.2.3

1 a 25 cm b 30 cm
c 14.1 cm d 11.4 cm

2 a 15 cm b 10.5 cm
c 3 cm d 24.6 cm

3 base, height (in cm) =
a 1, 3 b 1, 5 c 2, 7 d 5, 8

4 a 3 b 13 c 6 d 20

5 a 1.94 m. This assumes they are holding the ribbon at ground level. They can actually stand slightly further away than this, depending on the height at which they hold the ribbon above the ground, which we do not know.
b 10.8 km

6 a 22.2 cm b 5.77 cm

7 100 cm

Practice questions 4.2.4

1 29.7

2 a 5 b 10.3 c 7.43

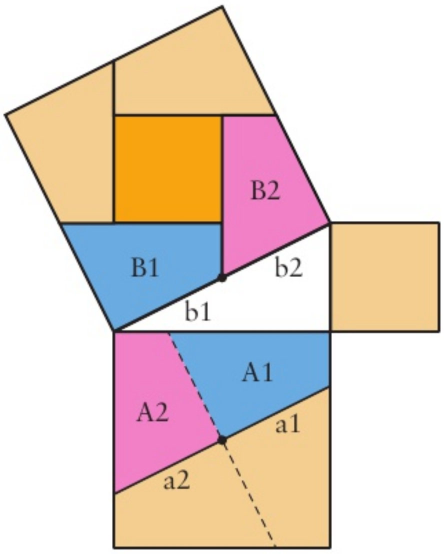
3 a 33.5 b 48.8

4 a right-angled
b right-angled
c non right-angled

Practice questions 4.3.1

1 a $2a + b$; $a + 2b$; $(2a + b)(a + 2b)$;
$2a^2 + 5ab + 2b^2$
b 10; $\frac{1}{2}ab$; $2a^2 + 5ab + 2b^2 - 10 \times \frac{1}{2}ab$;
$2a^2 + 2b^2$
c The green area, c^2, is equal to the total, coloured area, $2a^2 + 2b^2$, take away the orange and blue areas, $a^2 + b^2$

2



The centre of the square is the midpoint of the line segments through the centre, so a1 = a2. Also, the point on the hypotenuse is its midpoint, so b1 = b2. The various parallel lines mean that the triangle together with quadrilaterals A1 and A2 form a parallelogram. Hence, a1 + a2 = b1 + b2, and a1 = b1, and a2 = b2.

The parallel lines make the corresponding angles of quadrilaterals A1 and B1 congruent, which means the two shapes are similar. Further, since a1 = b1, A1 and B1 are congruent shapes. Similarly, A2 and B2 are congruent shapes. Further, all four quadrilaterals in each of the squares will be congruent to one another.

Therefore, the longest side of A2 is congruent to the longest side of B2. The longest side of A2 is also congruent to the opposite side of the parallelogram mentioned earlier, which is formed by the short side of the right triangle and the shortest side of A1. The short side of the triangle is also the side of the small square on the right side of our diagram, and the short side of A1 is congruent to the short side of B1. The side of the orange square plus the short side of B1 equals the long side of B2, which equals A2, and the side of the square on the right of the diagram plus the short side of A1, which equals the short side of B1, gives the same length.

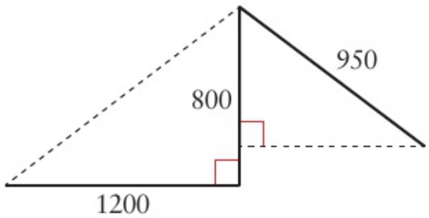
Hence, the height and area of the small square on the right equals the height and area of the orange square. Thus, the area of the square on the hypotenuse equals the sum of the areas of the squares on the legs.

3 a $(a + b)^2 - 4 \times \frac{1}{2}ab = a^2 + b^2$

b It proves the theorem, because the square is on the hypotenuse, and it is the sum of the areas of the squares on the legs of the right triangle.

Practice questions 4.3.2

1 a



b 1442 m c 672 m

2 399 m

3 35.7 m

4 2.83 m

5 a 47.9 in b 8.09 cm by 14.4 cm

6 15.4 cm

7 a 57 sq m b €179.55

8 a 763 m b 1126 m

9 9 cm², 21 cm²

10 a 356 cubits b 419 cubits

11 25 m

12 a 8.54 cm b 9.34 cm

c 29.35 cm²

Check your knowledge questions

1 Student's own paper triangles

2 a 13 b 10 c 5

d 4.80 e 8 f 14.1

3 a 12 b 6 c 4.12
d 1.67 e 4 f $\sqrt{3}$

4 10.3 cm or 103 mm

5 2.18 cm; 9.81 cm^2

6 $\sqrt{109}$

7 62.4 cm^2

8 85 cm

9 460 − 520 m

10 168 m

11 3.5 m

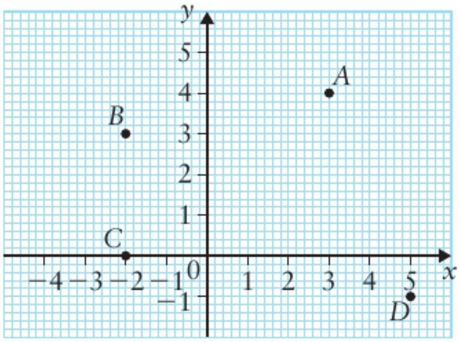
12 (25, 0), (24, 7), (20, 15), (15, 20), (7, 24), (0, 25), (−7, 24), (−15, 20), (−20, 15), (−24, 7), (−25, 0), (−24, −7), (−20, −15), (−15, −20), (−7, −24), (0, −25), (7, −24), (15, −20), (20, −15), (24, −7)

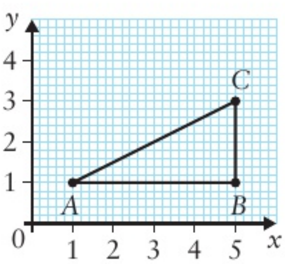
13 30.1 m

14 110 m

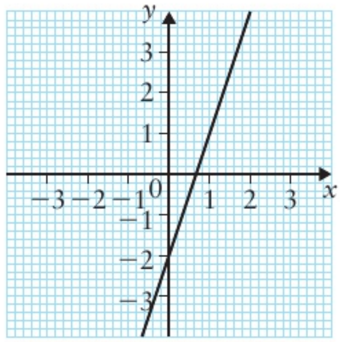
Chapter 5 answers

Do you recall?

1


2 a


b Assuming graph paper of 1 cm squared AB is 4 cm, BC is 2 cm and AC is approximately 4.5 cm.

3


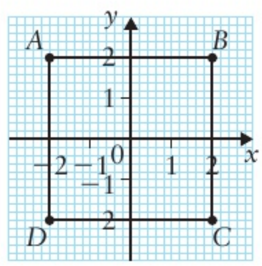
4 a $x = 13$ b $x = 2$

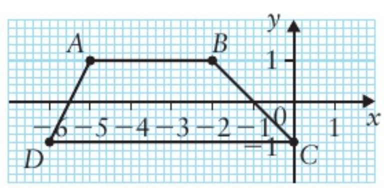
5 $x > 2$

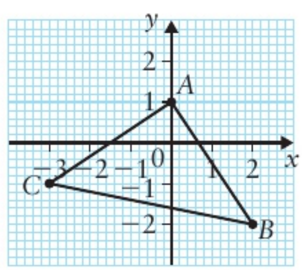

Practice Questions 5.1

1 a $A(-5,6)$ b $B(2,4)$
c $C(-4,-2)$ d $D(3,-1)$
e $E(1,1)$ f $F(3,1)$
g $G(-3,1)$ h $H(-1,-2)$
i $I(-2,-3)$ j $J(-2,2)$
k $K(-2,5)$

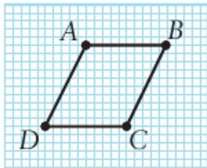
2 a Right-angled triangle with area 2 units squared.
b Trapezoid with area 10.5 units squared

3 a Square


b Trapezoid


c Triangle


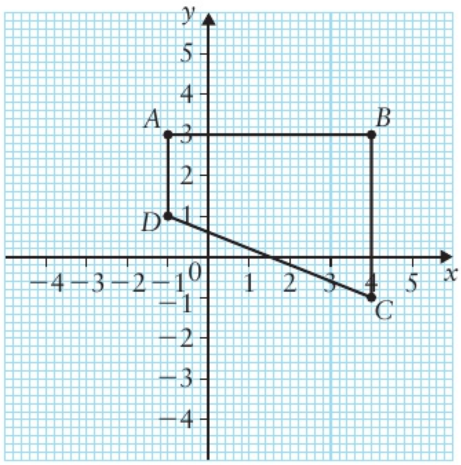
d Parallelogram



4 a 5 units b 6 units c 9 units

d 10 units e 5 units

5 a



b Area = 16 units squared c 17.1 units

6 a

x	−3	**0**	3	**6**
y	**4**	3	**2**	1

b

x	−3	**−2**	−1	**0**
y	**0**	2	**4**	6

c

x	1	**2**	−3	**−5**
y	**1**	2	**−3**	−5

d

x	−4	**any value**	0	**any value**
y	**−1**	−1	**−1**	−1

e

x	2	**2**	2	**2**
y	**any value**	0	**any value**	−4

7 a line AG b line CD

c line AB d line EF

Practice questions 5.2

1 a $y = 2x$

x	0	1	2
y	0	2	4

b $y = 3x + 1$

x	0	1	2
y	1	4	7

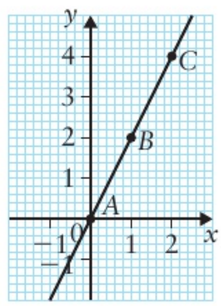
c $y = 2x - 1$

x	0	1	2
y	−1	1	3

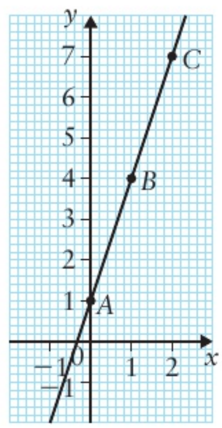
d $y = 2 - 3x$

x	0	1	2
y	2	−1	−4

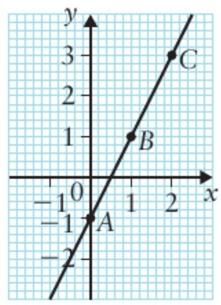
2 a



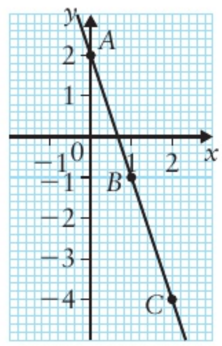
b



c



d



The lines are parallel under the graphs.

3 a $y = 2x$

x	0	0
y	0	0

b $y = 3x + 1$

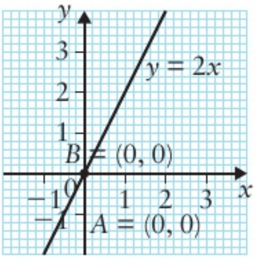
x	0	$-\frac{1}{3}$
y	1	0

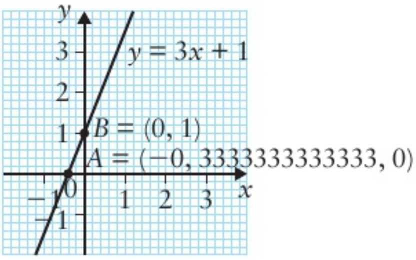
c $y = 2x - 1$

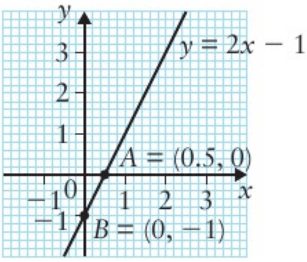
x	0	$\frac{1}{2}$
y	-1	0

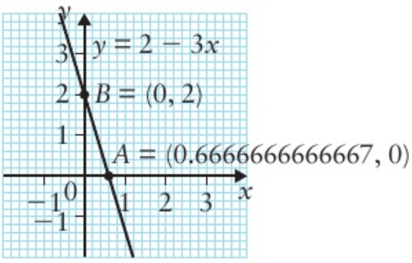
d $y = 2 - 3x$

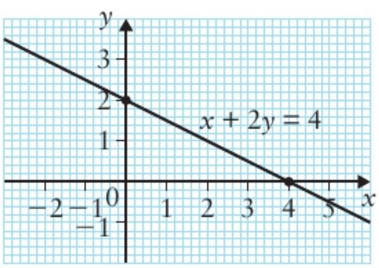
x	0	$\frac{2}{3}$
y	2	0

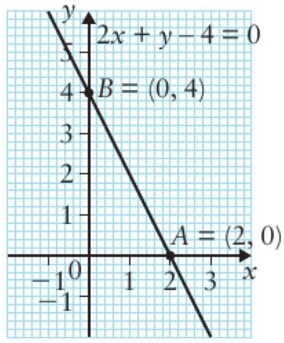
4 a


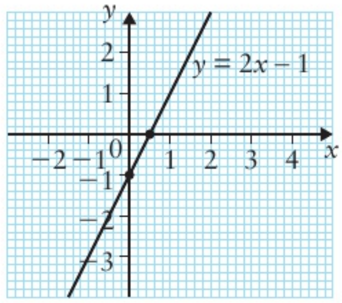
b


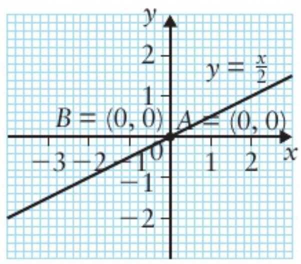
c


d


5 a


b


c


d


Line in part a is perpendicular to line in part c. Line in part b is perpendicular to line in part d.

6 b That is $(1,-2)$ lies on $x - 2y - 5 = 0$
Let $x = 1$ and solve for y in each equation:
$1 - 2y - 5 = 0 \Rightarrow -2y - 4 = 0 \Rightarrow y = -2$
and $y = -2$ in the coordinates
For the example which doesn't work:
$2 - 3y = 6 \Rightarrow -4 = 3y \Rightarrow y = \frac{-4}{3} \neq -2$

7 a $Y: y = \frac{1}{2}x + 1$ b $X: y = 2x - 1$
c $Z: y = -2x + 2$ d $W: y = -\frac{1}{2}x - 1$

8 Y: $(-2, 0)$ and $(0, 1)$ X: $(\frac{1}{2}, 0)$ and $(0, -1)$
Z: $(1, 0)$ and $(0, 2)$ W: $(-2, 0)$ and $(0, -1)$

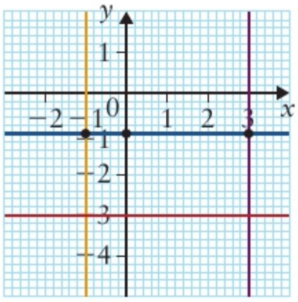
9 $(1, 4)$ lies on the line $2x - y + 2 = 0$;
$2.1 - 4 + 2 = 0$, $2 - 4 + 2 = 0$ true.
$(-2, -2)$ also lies on the line
$2.(-2) - (-2) + 2 = 0$, $-4 + 2 + 2 = 0$ true.

Practice questions 5.3

1 a $x = -2$ b $y = 4$

2 A: $x = 4$, B: $x = 2$, C: $x = -3$, D: $x = -0.5$, E: $y = 3$, F: $y = -2$, G: $y = 1$

3 It is a rectangle.



4 $(-3, 3)$, $(-0.5, 3)$, $(2, 3)$, $(4, 3)$, $(-3, 1)$, $(-0.5, 1)$, $(2, 1)$, $(4, 1)$, $(-3, -2)$, $(-0.5, -2)$, $(2, -2)$, $(4, -2)$. Total 12 points!

5 Point $(4, 2)$ lies on $x = 4$ line since $x = 4$ line has all possible y values on it. But $(4, 2)$ does not lie on the line $y = 3$. Both scenarios can be seen when they are plotted on a coordinate system.

6 a $(2, 5)$ b $(-3, 5)$ c $(-4, -3)$
d $(0, 2)$ e $(-5, 0)$ f $(0, 0)$

7 a $(2, -3)$ b $(2, 2)$ c $(6, -4)$
d $(-1, 1)$ e $(0, -4)$ f $(6, 0)$

Practice questions 5.4

1 a $m_{AB} = \frac{3(\text{rise})}{3(\text{run})} = 1$ b $m_{AB} = \frac{-3(\text{rise})}{2(\text{run})}$

c $m_{AB} = \frac{1(\text{rise})}{4(\text{run})}$ d $m_{AB} = \frac{-1(\text{rise})}{3(\text{run})}$

e $m_{AB} = \frac{2(\text{rise})}{3(\text{run})}$ f $m_{AB} = \frac{-2(\text{rise})}{2(\text{run})} = -1$

2 a $m_{AB} = \frac{1-3}{1-2} = \frac{-2}{-1} = 2y = 2x - 1$

b $m_{CD} = \frac{0 - -(1)}{-1-0} = \frac{1}{-1} = -1y = -x - 1$

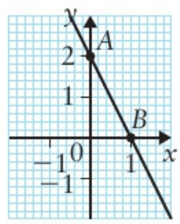
c $m_{EF} = \frac{1-4}{3-2} = \frac{-3}{1} = -3y = -3x + 10$

d $m_{GH} = \frac{1.5-3}{0-(-1.5)} = \frac{-1.5}{1.5} = -1y = -x + 1.5$

3 $m_{AB} = \frac{-2(\text{rise})}{2(\text{run})} = -1$ $m_{CD} = \frac{-2(\text{rise})}{2(run)} = -1$
$m_{AB} = m_{CD} = -1$ and the two lines are parallel.

4 a $m = 2, c = 5$ b $m = 1, c = -1$
c $m = -1, c = 3$ d $m = 5, c = -1$
e $m = 3, c = 0$
f m is undefined, no c and hence no y-intercept!
g $m = 0, c = 4$ h $m = \frac{3}{2}, c = 2$
i $m = 1, c = 1$ j $m = \frac{-3}{4}, c = -3$

5 a



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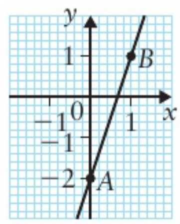

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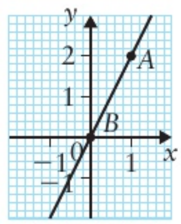
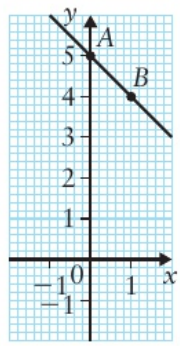
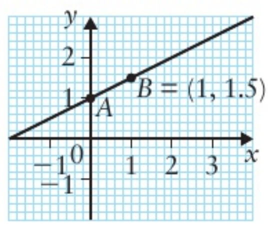


d



e



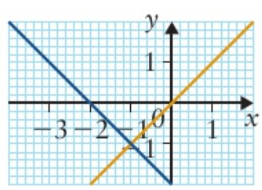
6 a Parallel lines, b Parallel lines,
c Not parallel lines, d Parallel lines

7 a Y b Z c V d S e T

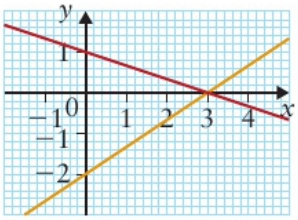
Practice questions 5.5

1 a $(1, 0)$ b $(2, -1)$
c $(-2, 2)$ d $(0, 0)$

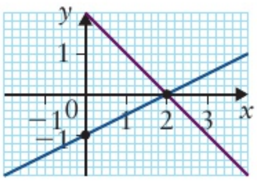
2 a $(-1, -1)$



b (3, 0)



c (2, 0)



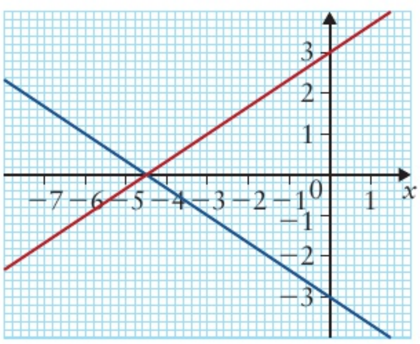
d No intersection, they are parallel!

3 **a** **i** (−3, 6) **ii** (1, 5) **iii** (−2, 2) **iv** (2, 1)

b *ABCD* is a parallelogram because opposite sides are equal and parallel.

c *AC* and *BD* are parallel (both equations' gradients are −4), therefore they will never meet.

4 (−4.5, 0) satisfy both equations of lines:
(1) 2. (−4.5) − 3.0 + 9 = 0, −9 − 0 + 9 = 0
Both equations are equal
(2) −2.(−4.5) − 3.0 − 9 = 0, 9 − 0 − 9 = 0
Both equations are equal. See the graphical solution below:



5 (0, 6) and (3, 0)

6 We need to write both equations in gradient intercept form to identify their gradients. Therefore, $x - 2y = 1$ can be written as $y = \frac{1}{2}x - \frac{1}{2}$ and $y = 2x + 1$. The product of the gradients is $\frac{1}{2} \times 2 = 1$. Thus, the two lines are not perpendicular!

7 If two lines are parallel, their gradients must be the same.
Line 1: $2x - 4y + 6 = 0$, $2x + 6 = 4y$,
$\frac{2}{4}x + \frac{6}{4} = \frac{4}{4}y$, $\frac{1}{2}x + \frac{3}{2} = y$, $m_1 = \frac{1}{2}$
(from gradient-intercept form).
Line 2: $y = \frac{1}{2}x - 1$ is already given in gradient-intercept form and $m_2 = \frac{1}{2}$.
Therefore $m_1 = m_2 = \frac{1}{2}$ and the two lines are parallel.

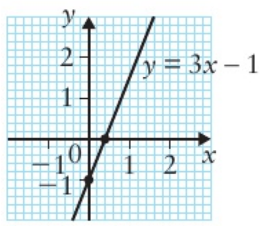
Check your knowledge questions

1 **a** *M* **b** *E* **c** *J* **d** C

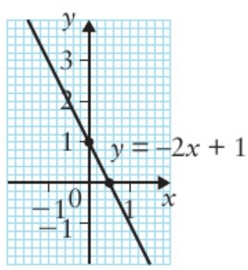
2 **a** (−1, 1) **b** (−2, −1)
c (4, 2) **d** (−2, 3)

3 **a** $MP = 5$ **b** $DF = 8$ **c** $HJ = 4$
d $BH = 4$ **e** $AF = 5(3^2 + 4^2 = 5^2)$

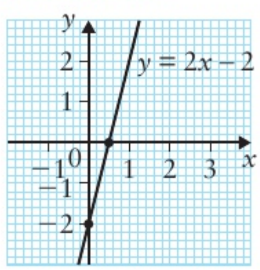
4 **a**

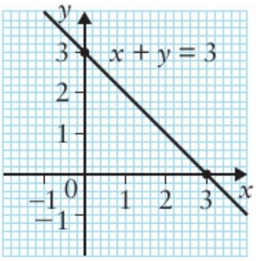


b



c



d 

5 a AE: $x = -2$ b AD: $y = 0$
c BC: $x = 0$ d EF: $y = -2$
e FG: $x = 1$

6 a $m_{AB} = \frac{1}{2}$ b $m_{AC} = 1$
c $m_{CD} = -1$ d $m_{BD} = -\frac{1}{2}$
e m_{FG} = undefined f $m_{EF} = 0$

7 a $y = -x - 2$ b $y = \frac{1}{2}x + 1$
c $y = -\frac{1}{2}x + 1$ d $y = -x + 2$
e $y = \frac{1}{2}x - 1$

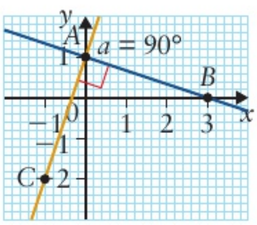
8 a $(0, -2)$ b $(0, 1)$
c $(0, 0)$ d No intersection
e No intersection

9 a $(\frac{5}{3}, 0)$ and $(0, -5)$ b $(1, 0)$ and $(0, 1)$
c $(5, 0)$ and $(0, -5)$ d $(6, 0)$ and $(0, 4)$

10 $m_{AB} = \frac{2-3}{1-0} = \frac{-1}{1} = -1$, y intercept is 3 (c), therefore the equation of the line is $y = -x + 3$

11 a $(-0.5, 0)$ b $(0, 1)$
c $m_{AB} = \frac{4}{2} = 2$ $y = 2x + 1$

12 a i $y = -\frac{1}{3}x + 1$ ii $y = 3x + 1$
b $m_{AB} . m_{AC} = -\frac{1}{3} . 3 = -1$, therefore AB and AC are perpendicular lines. They have the same y intcept. See the below illustration as well:



13 a $2 = 2.1 - 1$ is not true, so $A(1, 2)$ is not on $y = 2x - 1$
b $-1 + 1 = 0$ is true, so $B(-1, 1)$ is on $x + y = 0$
c $2 - 2.(-1) + 5 = 0$ is not true, therefore $C(2, -1)$ is not on $x - 2y + 5 = 0$
d $2. 0 - 5. 1 = -5$ is true, thus D $(0, 1)$ is on $2x - 5y = -5$

Chapter 6 answers

Do you recall?

1 $a = 7$

2 no

3 a $\frac{31}{108}$ b $2x^2 + 5x^2y - 5xy^2 + 7$
c 7

Practice questions 6.1

1 $a = 1$
2 $p = 4$
3 $y = 2$
4 $s = -2$
5 $n = -5$
6 $d = 3$
7 $m = -\frac{3}{4}$
8 $k = -2$
9 $c = 7$
10 $v = -4.8$
11 $u = -2$
12 $y = -3$
13 $j = \frac{5}{8}$
14 $z = -4$
15 Width = 100 cm; Length = 500 cm
16 Alex has 20 friends.
17 Jiwon spent 16 minutes on the homework assignment, Paul spent 32 minutes on the homework assignment and Dominic spent 64 minutes on the homework assignment.
18 Each box of cereal cost £2.
19 Sebastian has €17 and Floriane has €37.
20 The history book has 175 pages, and the book of poems has 75 pages.
21 The number is 26
22 The electrician worked for 5.5 hours.

23 The three consecutive odd integers are −51, −49 and −47

24 It will take 78.125 hours or 78 hours and 7.5 minutes for both pumps working together to fill all five pools.

Practice questions 6.2

1 $a = -7$

2 $r = 2$

3 $t = 17$

4 $n = \frac{1}{2}$

5 $p = -\frac{1}{4}$

6 $l = -3$

7 $k = -\frac{2}{3}$

8 $w = -13$

9 $q = 7$

10 $z = -3$

11 $g = 2.5$

12 $y = -10$

13 $b = 0$

14 $v = \frac{8}{3}$

15 $y = 4$

16 $x = 8$

17 $d = -\frac{4}{3}$

18 $t = -12$

19 $7 + 6n = 3n - 11$ so $n = -6$

20 a Fitness4U is cheaper for 10 classes per month.

b BFit is cheaper for 20 classes per month.

c $135 + 5c = 90 + 8c$

21 It will take 5 weeks.

22 $x = 4$. The side lengths of the triangle are 9, 17 and 8. The side lengths of the trapezoid are 5, 7, 6 and 16

23 The width is 2 and the length is 8

24 The perimeter is 87

25 The cost will be the same for 10 hours of work.

26 $2n - 13 = 5n + 14$ so $n = -9$

27 After 10 days, they will have the same amount of money.

28 Selma must sell 25 dozen sweetcorn cobs per day to break even.

Practice questions 6.3

1 $t = 6$

2 $a = 3$

3 $k = -4$

4 $a = 5$

5 $h = \frac{5}{3}$

6 $m = 8$

7 $d = \frac{7}{2}$

8 $n = 2$

9 $q = -\frac{9}{5}$

10 $f = -36$

11 $h = -4$

12 $g = \frac{3}{5}$

13 $w = 0$

14 $z = -\frac{11}{4}$

15 $p = 56$

16 $v = 4.5$

17 $y = -\frac{1}{8}$

18 $x = \frac{36}{7}$

19 $4(2n - 7) = -9$ The number is $\frac{19}{8}$

20 $3(5n + 12) = 8 - 4n$ The number is $-\frac{28}{19}$

21 The perimeter is 14

22 a Rectangle A: width = 4 and length = 21
Rectangle B: width = 3 and length = 28

b Area = 84

23 It will take just over 117 minutes

24 It will take 8.8 months or 9 full months.

25 It will take 0.625 hours or 37.5 minutes

26 There are 4 rabbits, 16 cats and 40 dogs

27 The fishing boat was travelling at 31 km/hr and the cruise ship was travelling at 43 km/hr

28 Max invested £6800 in the account earning 6.5% simple interest and £13 200 in the account earning 8% simple interest.

Practice questions 6.4

1 $h = -24$

2 $n = 10$

3 $w = -4$

4 $m = -2$

5 $f = 21$

6 $p = -4$

7 $x = -13$

8 $q = -15$

9 $a = -48$

10 $g = 4$

11 $h = 2$

12 $t = 8$

13 $y = -40$

14 $f = 11$

15 $n = -7$

16 $p = 12$

17 $b = 9$

18 $c = 21$

19 $g = -\frac{1}{2}$

20 $y = 8.5$

21 $\frac{7n}{8} = 3 + \frac{5n}{6}$; $n = 72$

22 $\frac{n + 12}{(n + 5) - 1} = \frac{5}{3}$ The original fraction is $\frac{8}{13}$

23 Mary had \$30 to start with.

24 a Original strike percentage is 63.6%

b He needs to throw an additional 70 consecutive strikes.

25 Emily, Tara and Rita can complete the set in 4 hours if working together.

26 a First cyclist: $25 = (r + 4.5)t$
Second cyclist: $21 = rt$

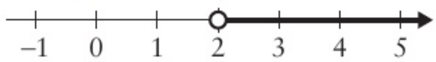
b It took them 53.3 minutes or 53 minutes.

27 $\frac{1}{3}(8n - 5) = 2n + 7$ The number is 13.

28 185 units were produced.

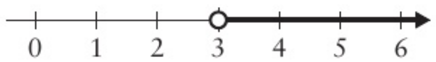
29 In the year 2010, the average price was \$10.70

Practice questions 6.5

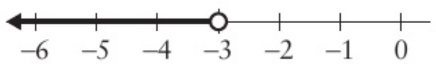
1 

2 

3 

4 

5 

6 

7 

8 

9 $x < -7$

10 $x \geq 3$

11 $x > -2$

12 $x \leq 10$

13 $x < -13$

Practice questions 6.6

1 $y \geq 3$

2 $a < 5$

3 $h < 4$

4 $c \leq -3$

5 $b < -18$

6 $n < 5$

7 $q \leq 4.75$

8 $w \leq 3$

9 $f < 1$

10 $l < 3$

11 $z \geq -14$

12 $y \leq 21$

13 $r < 14$

14 $x \leq 5.5$

15 $d \leq -49$

16 $x > -11.6$

17 $n \leq 24.5$

18 $y \geq -\frac{10}{3}$

19 $x < 4$

20 $x > -\frac{1}{4}$

21 $5.50n - 124 > 0$ She must sell at least 23 scarves to break even.

22 $4(x - 60) < 125$ If the regular monthly fee is less than €91.25, then Hugo will join the gym.

23 $17 + 4n \leq 2n - 15$ The number is less than or equal to -16

24 $18 - 5n > n + 6$ The number is less than 2

25 average speed $= \frac{20 - 5 \text{ km}}{135 - 27.6 \text{ min}}$

Her average running speed should be more than 0.14 km/minute

26 $75 + 28.5h > 60 + 31.25h$ If BE Electrical works for less than 5.45 hours, they will be more expensive than 4U Electrical.

27 $\frac{1}{2} \times 6(4x - 2) \geq 66$ $x \geq 6$

28 $2(4w - 3) + 2w < 224$ The width must be less than 23 cm

Check your knowledge questions

1 $d = -3$

2 $f = 11$

3 $v = 4$

4 $b = -4.5$

5 $m = 8$

6 $a = 4$

7 $h = -6$

8 $x = -18$

9 $k = 3$

10 $u = 4$

11 $t = 1$

12 $x = 0$

13 $r = -2$

14 $y = 4$

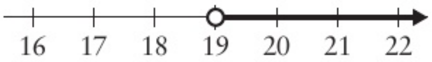
15 $p = -51$

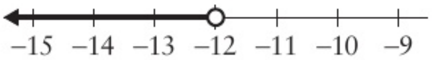
16 $n = 6$

17 $x > 8$

18 $x \geq -1$

19 $w < 6.5$ **20** $j \geq 12$

21 

22 

23 $x \geq -17$ **24** $x < 8$

25 $2x + (2x + 2) + (2x + 4) + (2x + 6) = 364$
The consecutive even integers are 88, 90, 92 and 94

26 1.5 litres of sugar is needed, and 750 ml of vinegar is needed.

27 $(5.75 - 3.25)x = 350\,000$
They need to sell 140 000 boxes of tea in order to break even.

28 $35 + 12f \leq 250$
Ghita can bring at most 17 friends.

29 $100 \times \frac{5}{2} = s \times (\frac{5}{2} + \frac{20}{60})$
Tarek's average speed was about 88 km/hr

30 $6(x + 4) = 7(2x - 7)$
Perimeter = 78.75 + 78.75 + 38 = 195.5 mm

31 **a** The new solution has 43.3% glycerine

b $\frac{1}{3+x}\left(\frac{50x}{100} + \frac{3 \times 40}{100}\right) = \frac{45}{100}$
3 litres of the 50% solution is needed.

32 **a** It will cost approximately 1105 thousand dollars to remove 25% of the pollutant.

b It will cost 30 thousand dollars to remove 50% of the pollutant.

c They could remove 80% of the pollutant.

Chapter 7 answers

Do you recall?

1 True: Supplementary angles add up to 180°

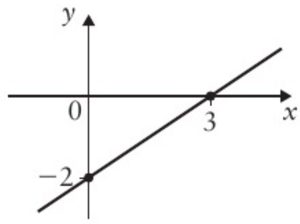
2 False: Complementary angles add up to 90°
43° + 57° = 100°

3 True: The angles in a triangle add up to 180°

4 True: The angles in any quadrilateral add up to 360°

5 30°

6 30°, 30°, 120°

7 

8 108 cm squared

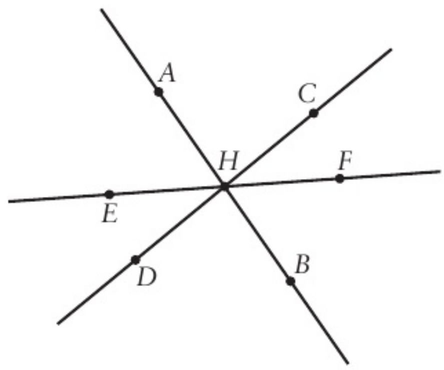
Practice questions 7.1

1 **a** $79° + x° = 90°, x° = 11°$

b $15° + y° + 37° = 90°, y° = 38°$

c $45° + 97° + z° = 180°, z° = 38°$

d $d° + 38° = 180°, d° = 142°$

2 

6 possible adjacent angles are; ∢AHC and ∢CHF, ∢CHF and ∢FHB, ∢FHB and ∢BHD, ∢BHD and ∢DHE, ∢DHE and ∢EHA, ∢EHA and ∢AHC

3 **a** $2x° + 3x° = 180°, x° = 36°$

b $y° + 3y° + 5y° = 180°, y = 20°$

c $z° + 90° + 2z° = 180°, z° = 30°$

d $2k° + 28° = 90°, k° = 31°$

4 **a** $b° = 80°$ (vertically opposite angles)

b $y° = 50°$ (vertically opposite angles), $90° + x° + 50° = 180°$ (straight angle/supplementary angles), thus $x° = 40°$, $z° = 90° + x° = 130°$ (vertically opposite angles)

c $2z° = 120°$ (vertically opposite angles), thus $z° = 60°$, $120° + 3x° = 180°$ (supplementary angles), $x° = 20°$, $y° = 3x° = 60°$ (vertically opposite angles) d $x° + 2x° + 3x° + 4x° = 360°$ (revolutionary angles), $10x° = 360°$, thus $x° = 36°$

5 a $x° = 65°$ (corresponding angles), $x° = y° = 65°$ (vertically opposite angles)

b $x° = 70°$ (alternate interior angles), $y° = 70°$ (vertically opposite angles)

c $87° + y° = 180°$ (co-interior angles), $y° = 93°$, $y° = x° = 93°$ (corresponding angles)

d $x° = 55°$ (alternate interior angles), $55° + y° = 180°$ (co-interior or supplementary angles), $y° = 125°$

6 a $2x° + 64° = 180°$ (co-interior), $2x° = 116°$, $x° = 58°$, $4y° = 64°$ (corresponding), $y° = 16°$

b $4x° = 116°$ (alternate-exterior), $x° = 29°$, $2y° + 116° = 180°$ (supplementary), $2y° = 64°$, $y° = 32°$

c $9x° = 126°$ (corresponding), $x° = 14°$, $126° + 2y° = 180°$ (co-interior), $2y° = 54°$, $y° = 27°$

d $2y° + 124° = 180$ (supplementary), $2y° = 56°$, $y° = 28°$
$2y° = 4x° = 56°$, $x° = 14°$

7 $j° = 55°$ (vertically opposite), $55° = d°$(alternate-exterior), $d° = b° = 55°$(vertically opposite), $f° + 55° = 180°$(supplementary), $f° = 125°$, $f° = h° = 125°$(vertically opposite), $f° = c° = 125$ (alternate-interior), $f° = e° + 90° = 125°$ (corresponding), thus $e° = 35°$

Practice questions 7.2

1 a $(5 - 2).180° = 3.180° = 540°$

b $(9 - 2).180° = 1260°$

c $(14 - 2).180° = 2160°$

2 a $\dfrac{540°}{5} = 108°$ b $\dfrac{900°}{7} = 128.57°$

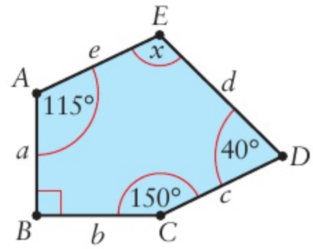
c $\dfrac{360°}{4} = 90°$ d $\dfrac{1440°}{10} = 144°$

3 a $\dfrac{360}{40}$ = 9-sided polygon; the sum of interior angles $(9 - 2).180° = 1260°$

b $\dfrac{360}{60}$ = 6-sided polygon; the sum of interior angles $(6 - 2).180° = 720°$

c $\dfrac{360}{45}$ = 8-sided polygon; the sum of interior angles $(8 - 2).180° = 1080°$

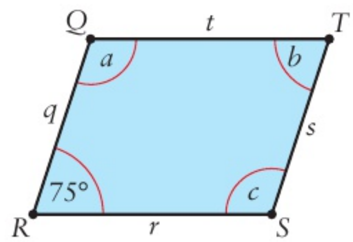
4 a *ABCDE* is a pentagon and the sum of the interior angles is 540°. Therefore, $115° + 90° + 150° + 40° + x° = 540°$, $x° = 145°$



b *FGHIJK* is a regular hexagon and the sum of the interior angles is 720°.
$y° = \dfrac{720}{6} = 120°$

c *LMNOP* is a pentagon.
So, $60° + z° + 90° + 90° + z° = 540°$, $240° + 2z° = 540°$, $z° = 150°$

d (Hint: QRST is a parallelogram)

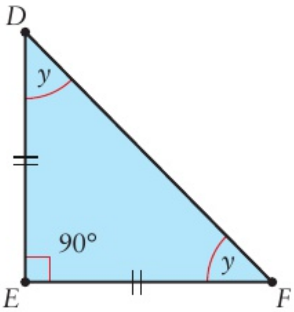


Since *QRST* is a parallelogram, opposite angles are equal, $b° = 75°$, $a° = c°$.
$b°$ and $c°$ are co-interior angles and they are supplementary. Therefore, $c° = 180° - 75° = 105°$, $a° = 105°$

5 a $x° + x° + x° = 180°$, $3x° = 180°$, $x° = 60°$

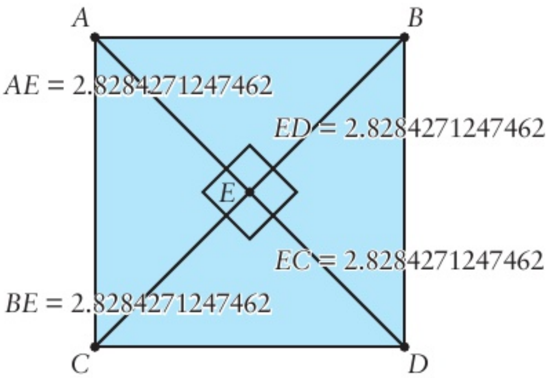
b $x° + 155° = 180°$, $x° = 25°$
$80° + x° + y° = 180°$, $80° + 25° + y° = 180°$, $105° + y° = 180°$, $y° = 75°$

c $90° + y° + y° = 180°, 2y° = 90°, y° = 45°$



6 a T b F c T d F
e T f T g F h T

7 We can show that by using Geogebra program;



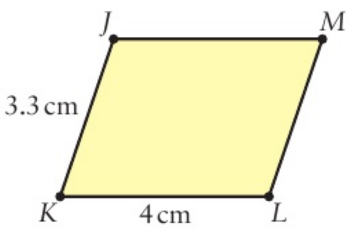
8 a $a = 20$
$2a + 4a = 120$

b $x = 110$
Angle $BDA = 70°$ isosceles triangle
$x = 180 - 70 = 110$ – angles along a straight line add up to 180

Practice questions 7.3

1 a $3 + 8 + 10 = 21$ cm

b $2.(4 + 3.3) = 2.\ 7.3 = 14.6$ cm



c $6.3 = 18$ cm

d $5 + 2 + 7 + \sqrt{8} = (14 + \sqrt{8})$ cm or $(14 + 2\sqrt{2})$ units

2 $2.(a + b) = 48$ cm, $a + b = 24$ cm, so possible three rectangles are 10 by 14 or 11 by 13 or 8 by 16 (any other combination with the sum of 24 cm is acceptable!)

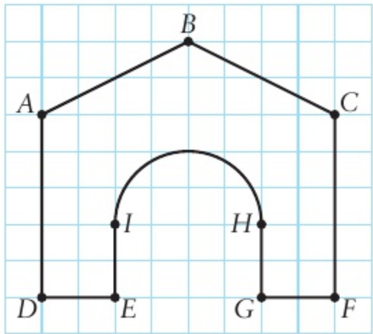
3 a $3\text{ cm} + 4\text{ cm} + 3\text{ cm} + \frac{4\pi}{2}\text{ cm} = (10 + 2\pi)$ cm

b When we combine 4 semi-circles it makes two full circle circumferences; 4π

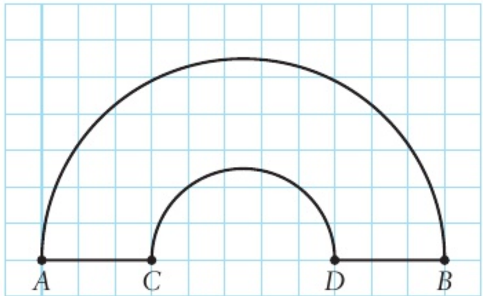
c $2\pi + 1 + 3 + \pi + 3 + 1 = (3\pi + 8)$ cm

d $14 + \frac{120°}{360°}.2\pi.7 = (14 + \frac{14}{3}\pi)$ cm

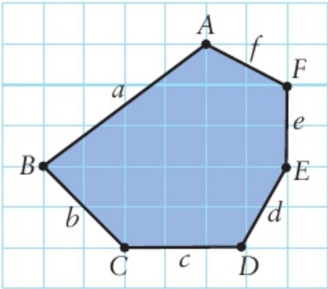
4 a $AB = BC = \sqrt{20}$ cm, JH semi-circle is 2π cm. If we add all the sides of the shape; $(2\sqrt{20} + 2\pi + 18)$ cm



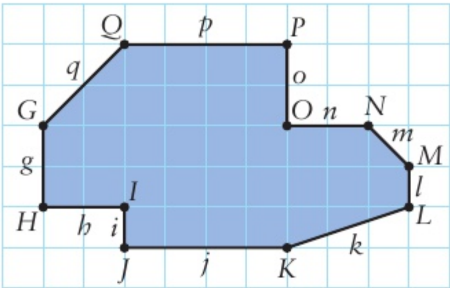
b Semi-circles in order are 5.5π and 2.5π, so the total perimeter of the shape is $5.5\pi + 2.5\pi + 3 + 3 = (8\pi + 6)$ cm



c $a = 5$ cm, $b = 2\sqrt{2}$ cm, $f = d = \sqrt{5}$ cm, $c = 3$ cm, $e = 2$ cm
So, $P = (10 + 2\sqrt{2} + 2\sqrt{5})$ cm



d $q = 2\sqrt{2}$ cm, $m = \sqrt{2}$ cm, $k = \sqrt{10}$ cm,
So, $P = (18 + 3\sqrt{2} + \sqrt{10})$ cm



5 a 327 m b 38 posts
c €323 for posts
€310.65 for wire
€633.65 total

6 a 28.38 b 59.4 c 112.8

7 a $(9\pi + 42)$ cm b $(14\pi + 52)$ m
c $(4.8\pi + 20.4)$ m d 64π mm

8 a 120 cm b 94 cm c 100 cm d 72 cm

9 a 76.0 cm b 116.1 cm

Practice questions 7.4

1 a $25\,\text{cm}^2$ b $121\,\text{mm}^2$
c $49\,\text{mm}^2$ d x^2 unit^2

2 a $60\,\text{cm}^2$ b $63\,\text{mm}^2$
c $384\,\text{m}^2$ d $1200\,\text{cm}^2$ or $0.12\,\text{m}^2$

3 a $4\pi\,\text{cm}^2$ b $25\pi\,\text{m}^2$
c $12.25\pi\,\text{mm}^2$ d $64\pi\,\text{cm}^2$

4 a 10 unit^2 b 5.04 unit^2
c 16.5 unit^2 d 18 unit^2
e 270.4 m f 320 m

5 a $83\,\text{cm}^2$ b $164\,\text{cm}^2$
c $103\,\text{cm}^2$

6 a $24\,\text{m}^2$
b $26\,\text{m}^2$
c 1100 bricks to nearest hundred

7 \$1500

Practice questions 7.5

1 SA
$= \frac{1}{2} \times 2 \times 2 + \frac{1}{2} \times 2 \times 2 + 2 \times 3 + 2 \times 3 + 3 \times \sqrt{8}$
$= 16 + 6\sqrt{2}\,\text{cm}^2$

2 $SA = 6 \times 4 + 2 \times 6 \times 3 + 2 \times 3 \times 4 = 108\,\text{cm}^2$

3 We have 8 cubes. The surface area of 8 cubes is $8 \times 6 \times 1\,\text{cm}^2 = 48\,\text{cm}^2$. But there are 4 common faces $4 \times 1\,\text{cm}^2 = 4\,\text{cm}^2$.
Therefore, the surface area of the shape is $48\,\text{cm}^2 - 4\,\text{cm}^2 = 44\,\text{cm}^2$

4 $SA = 2.\pi.2.3 + 2.\pi.2^2 = 20\pi\,\text{cm}^2$

5 $SA = 50 \times 25 + 2 \times 50 \times 2 + 2 \times 25 \times 2$
$= 1550\,\text{m}^2$
Total cost $= 1550\,\text{m}^2 \times 35€ = €54\,250$

6 a $356\,\text{cm}^2$ b $4740\,\text{cm}^2$ c $3572\,\text{cm}^2$

7 $363.08\,\text{m}^2$

8 $203\,\text{m}^2$

Check your knowledge questions

1 a $45° + x° = 180°$, $x° = 135°$
b $14° + y° = 90°$, $y° = 76°$

2 a $55° + 70° + 90° + x° = 360°$, $x° = 145°$
b $135° + 115° + 23° + z° = 360°$, $z° = 87°$

3 a $b° = 41°$(vertically opposite),
$41° + a° = 180°$, $a° = 139° = c°$
(vertically opposite angles)
b $14° = e°$(vertically opposite) ,
$f° + 14° = 180°$(angles on a straight line),
$f° = 166° = d°$

4 a $x° = 45°$ (vertically opposite),
$z° = 45°$(alternate interior),
$45° + y° = 180°$ (straight angles) $y° = 135°$

b $108° = m°$ (corresponding), $108° + n° = 180°$ (supplementary), $n° = 72°$, $n° = 72° = k°$ (alternate)

5 a $(5 - 2).180° = 3.180° = 540°$

b $360°$

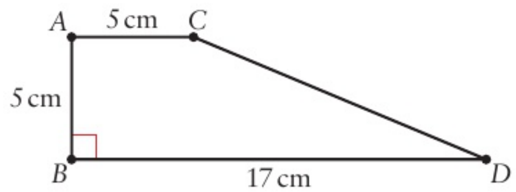
c $(n - 2).180° = 1260°$, $n - 2 = \frac{1260}{180} = 7$, $n = 9$, One of the exterior angles is $\frac{360}{9} = 40°$

d If one of the exterior angles is $36°$, then $n = \frac{360}{36} = 10$ number of sides.

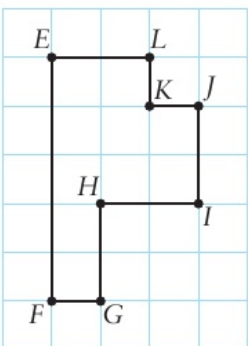
6 a $C = (2\pi.2.5) = 5\pi$ cm

b $D = 16$ m thus r = 8 m, $C = 2.\ \pi.8 = 16\pi$ m

7 a $CD^2 = 5^2 + 12^2 = 13^2$, $CD = 13$ cm, $P = 13 + 5 + 5 + 17 = 40$ cm



b $P = 5 + 1 + 2 + 2 + 2 + 1 + 1 + 2 = 16$ cm



8 a $3 + 3 + \frac{3}{4} \times 2.\pi.3 = \left(6 + \frac{9}{2}\pi\right)$ cm

b $4 + 4 + \frac{1}{4} \times 2.\pi.4 = (8 + 2\pi)$ cm

9 a $5 + 5 + 2\pi.2 = (10 + 4\pi)$ cm

b $5 + 3 + \frac{1}{2}.(3\pi) + 5 + \sqrt{10} = \left(13 + \frac{3}{2}\pi + \sqrt{10}\right)$

10 a $A = (8 + 4\pi)$ unit^2

b $A = 15 + 5 = 20$ unit^2

11 a $A = \frac{3}{4}\pi.3^2 = \frac{27}{4}\pi$ cm^2

b $A = \frac{1}{4}\pi.4^2 = 4\pi$ cm^2

12 a $A = \frac{1}{2}.5.(17 - 5) + 25 = 55$ cm^2

b $A = 10$ unit^2

13 a $A = \frac{1}{2}.3.\ (3 + 5) = 12$ unit^2

b $A = 3.\ 5 = 15$ unit^2

14 a $SA = 6.\ 4^2 = 96$ cm^2

b $SA = 2.\ \frac{4.3}{2} + 6.3 + 6.5 + 6.4 = 84$ cm^2

15 a $SA = 2(12.8 + 8.3 + 12.3) - 7.4 + 7.4 + 2(7.4 + 4.4) = 400$ cm^2

b $SA = 2(11.8 + 11.8) + 2(11.11 - 5.5) + 4.5.8 = 704$ cm^2

Chapter 8 answers

Do you recall?

1 CD is the perpendicular bisector of segment AB.

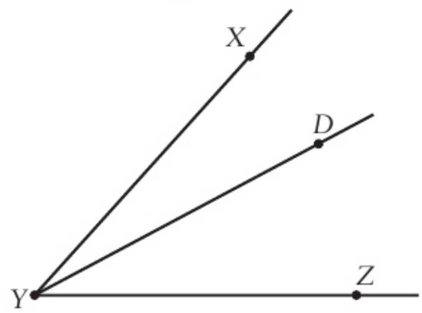
2 The ray BF is the bisector of the angle ABC.

Practice questions 8.1

1 To be marked by the teacher

2 To be marked by the teacher

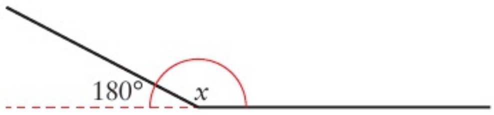
3 Construct angle $\sphericalangle XYZ$ to be congruent to $\sphericalangle A$ and using as common line YX, construct angle $\sphericalangle DYX$ to be congruent to $\sphericalangle B$. Then, the subtracted angle will be $\sphericalangle DYZ$



4 a To be marked by the teacher

b To be marked by the teacher

c Construct an angle congruent to x. Then, extend one of the two sides of x with the ruler, as shown below. The angle formed by the extended line and one of the sides of angle x is $180° - x$. Otherwise, you can subtract x from an angle of size $180°$.

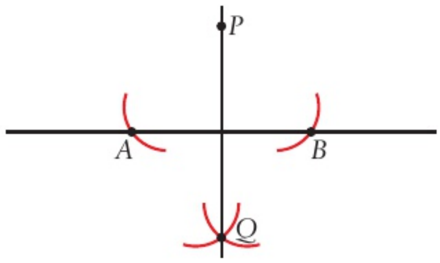


5 To be marked by the teacher

6 a Construct a 60°angle using the equilateral triangle construction of the Worked Example 8.3. A 120° angle is double a 60° angle. Add two congruent angles of size 60°

b A 240° angle is double a 120° angle. Add two congruent angles of size 120°. Otherwise, construct a 60° angle and add to it to a straight angle.

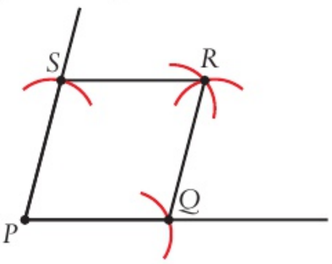
c A 90° angle can be obtained through the perpendicular lines constructions. One way is to follow the steps of the Investigation. Another way is the following:
Given a line *l* and a point *P* not on it, use any radius larger than the distance from *P* to *l* and draw two arcs with centre *P* that intersect line *l* in *A* and *B*. Use the same radius *PA* and point the compasses in *A* Draw an arc below the line. Point the compass in *B* and with the same radius *AP* draw an arc below the line that intersects the previous arc in *Q*. Join *P* to *Q*. Line PQ is perpendicular to line *l*



d A 150° angle is given by the addition of a 90° angle to a 60° angle. Use the angle addition construction.

7 To be marked by the teacher.

8 a Construct an angle congruent to angle *A* and name the new vertex *P*. Point the compasses in *P* and with radius *AB* draw an arc that intersects one of the sides of the angle *P*. Name the intersection *Q*. Point the compasses in *P* and with radius *AD* draw an arc that intersects the other sides of the angle *P*. Name the intersection *S*. Then, point the compasses in *S* and with radius *PQ* draw an arc. Point the compasses in *Q* and with radius *PS* draw an arc. Name the intersection *R*. *PQRS* is a parallelogram congruent to parallelogram *ABCD*.



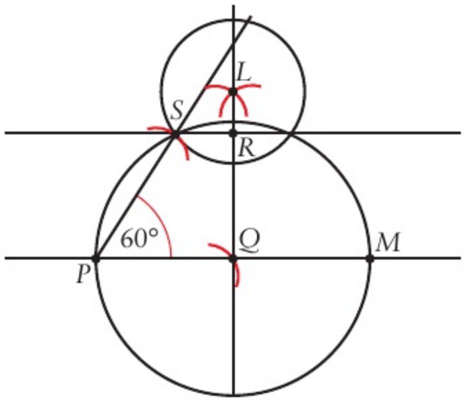
b Construct an angle congruent to a 60° angle using the equilateral triangle construction in Worked example 8.3. Use the steps of the Worked example 8.5 for parallelograms using the same radius of the compasses in every step.

c Step 1. construct an angle congruent to a 60° angle using the equilateral triangle construction in Worked example 8.3. Name the vertex *P*.

Step 2. Set the point the compasses at the vertex *P* of the angle and using the same radius draw two arcs on the sides of the angle. Name the intersections *Q* and *S*. This will ensure that the two adjacent sides *PS* and *PQ* are congruent.

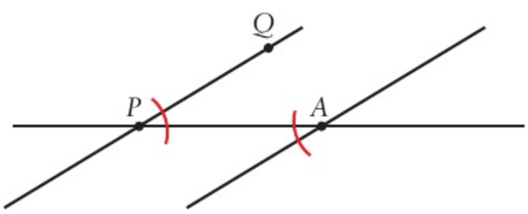
Step 3. Extend with a ruler the side *PQ*. Place the point of the compasses in *Q* with radius *QP* and draw two arcs that intersect the line *PQ* at *P* and at another point and name it *M*. Set the point of the compasses at *P* and draw an arc above the line *PQ*. Set the point of the compasses at *M* and draw an arc above the line *PQ* that intersect the other arc at a point. Name the intersection L. Join with a ruler Q to L. The side QL is perpendicular to QP.

Step 4. Follow the steps of the Investigation using S as point outside the line QL and two random points on the line QL and construct a perpendicular line going through S. The intersection of the perpendicular line through S and the line QL is point R. The figure $PQRS$ is the required trapezoid.



d In order to construct a square of side 4 cm: draw a 90° angle with vertex A. Using 4 cm as a radius follow the steps of the Worked Example 8.5 to construct a parallelogram.

9 Draw a line l and a point A not on l. Draw two arbitrary points on line l and name them P and Q. With a ruler draw line PA. Follow the steps of Worked Example 8.1 to copy an angle with vertex A congruent to angle QPA but in the position marked below:

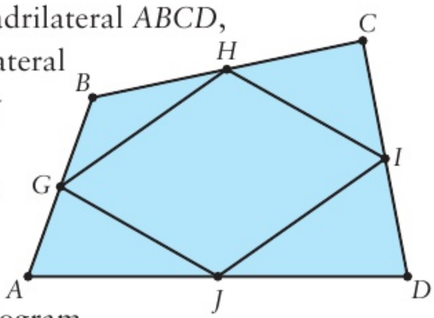


Practice questions 8.2

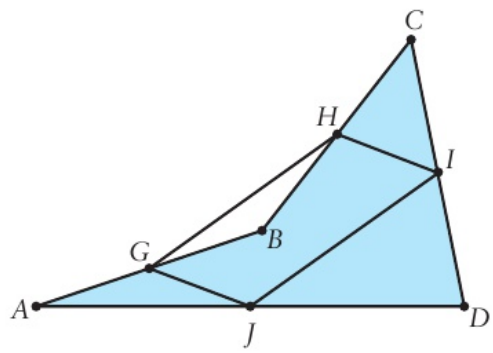
1 Follow the steps of the perpendicular bisector construction. To be marked by the teacher.

2 **a** The midpoint of a line segment can be constructed using the construction of the perpendicular bisector. To be marked by the teacher.

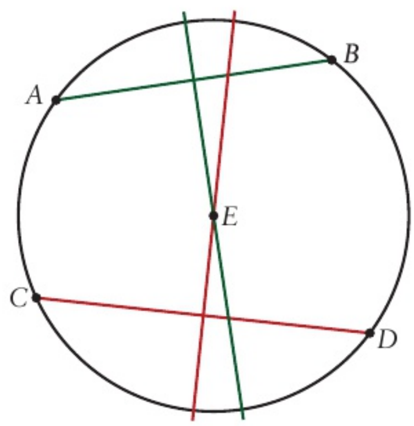
b Given a quadrilateral $ABCD$, the quadrilateral obtained by joining the consecutive midpoints of $ABCD$ is a parallelogram.



c Student's own diagrams including a concave quadrilateral.



3 Use the perpendicular bisector construction. The two perpendicular bisectors meet at one point, E, that is the centre of the circle. Indeed, the intersection of the two perpendicular bisectors has the property to be equidistant from the endpoints of both the two chords. This distance is the radius of the circle.



4 **a** Draw the perpendicular bisectors of *AB* and *BC*. They will meet at a point equidistant from *A*, *B* and *C*.

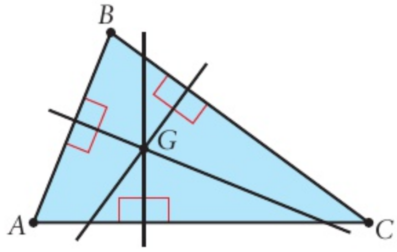
b This is the centre of a circle.

c The points cannot be collinear.

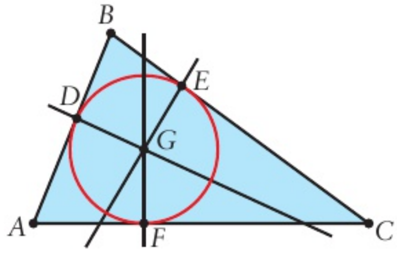
5 Follow instructions in parts a, b and c.

a The incircle is tangent to the sides of the triangle.

c The angle bisector of an angle is the locus of points equidistant from the sides of the angle. Therefore, the intersection of all the angles bisectors, G, of a triangle has the property to be equidistant from the sides.



The length of the perpendicular segments from the incentre to the sides of the triangle is the radius of a circle that passes through *D*, *E* and F.



6 **a** Rhombus and square

b Parallel

c Rhombus and square

d In general, diagonals and angle bisectors do not coincide in parallelograms. In rhombuses and squares the diagonals are also angle bisectors.

7 **a** To be marked by the teacher

b To be marked by the teacher

c The measure of *CAB* is half the measure of *COB*

d The angles at the circumference are half the size of the corresponding angles at the centre.

8 **a** To be marked by the teacher

b Two sides of an isosceles triangle are congruent. Therefore, the vertex *C* must be a point of the perpendicular bisector since it is equidistant from *A* and *B*.

c The angle bisector $\sphericalangle C$ is equal to the perpendicular bisector of the base.

9 The student's own paper creations

a Overlap *A* on *B*. The crease is the perpendicular bisector

b Overlap the side *BA* on *BC*. The crease is the angle bisector.

c Overlap the side *AB* on *DC* and *BC* on *DA*.

d Overlap two opposite vertices of the square. This triangle is right-angled and isosceles and has an area half of that of the square. Overlap the two vertices of the base of the isosceles triangle and get a triangle of area $\frac{1}{4}$ of the square.

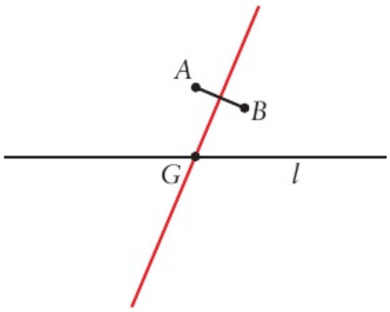
10 **a** Constructions to be marked by the teacher. *E* is called the orthocentre (the intersection of the three heights).

b Construction to be marked by the teacher. In an isosceles triangle the orthocentre, incentre and circumcentre are all on the same line.

11 **a** It is the locus of all the points and only those that are equidistant from *A* and *B* and at a distance less than or equal to 4 cm from *M*.

b It is the locus of all the points and only those that are at a distance of less than 4 cm from A and also belong to the equilateral triangle ABC of side 6 cm.

12 The locus of points in this case is just one point: the intersection G between the line l and the perpendicular bisector.



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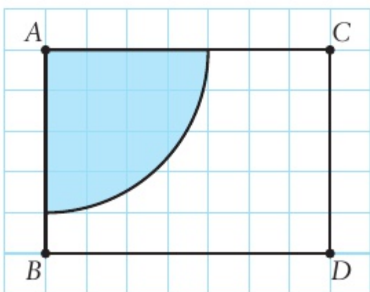
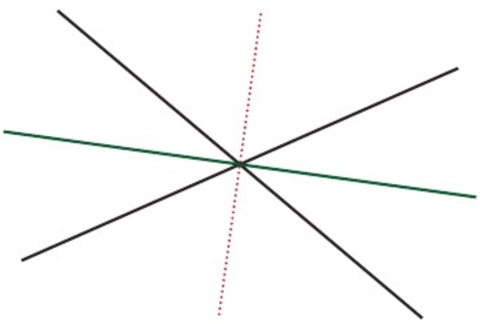


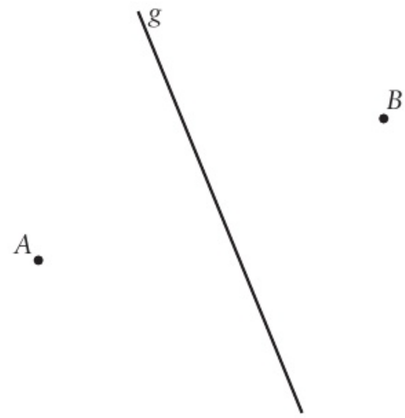
14 The locus is the union of 4 perpendicular bisectors (4 rays) of the 4 corresponding angles



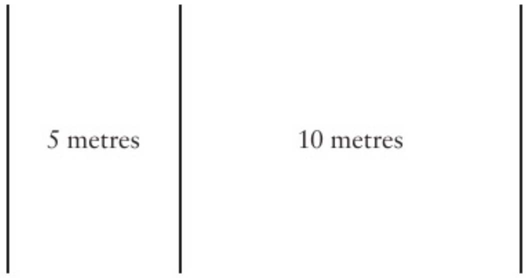
15 The locus is the parallel line equidistant from the two lines.

16 The locus contains the two parallel lines to l, that are at a distance 3 cm from l together with semi-circles of radius 3 cm at each end of the line l.

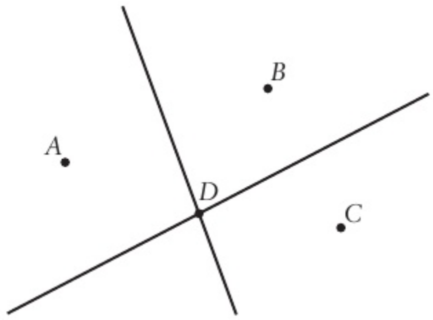
17 Line g is the freeway. (perpendicular bisector of AB).



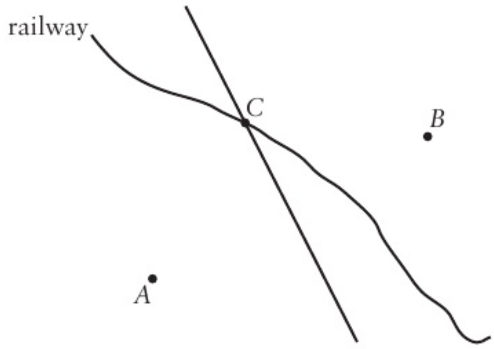
18



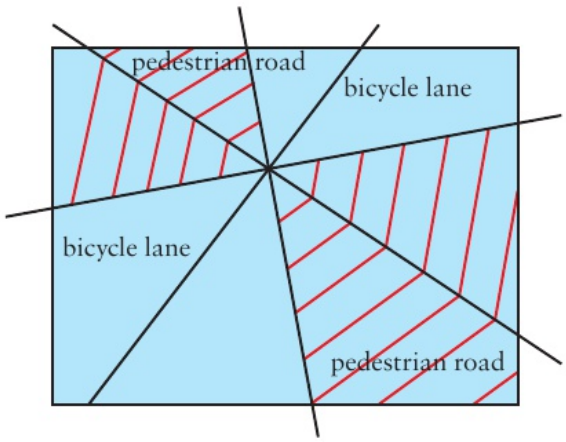
19 The fourth post office, D, is at the intersection of the perpendicular bisectors of BC and AB.



20 The station is at the intersection of the perpendicular bisector of AB and the railway.



21 The 4 angle bisectors determine 8 regions. The zebra regions below are closer to the pedestrian road than the bicycle lane



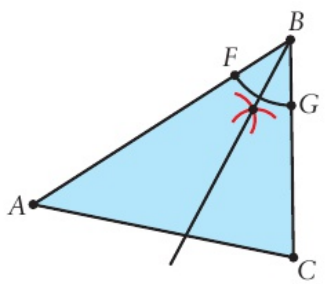
Check your knowledge questions

1 The true statements are:

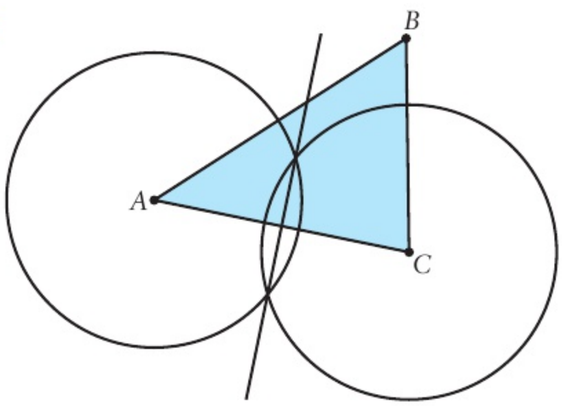
Ruler and compasses constructions are based on theorems and properties and are rigorous processes.

Ruler and compasses constructions use ruler and compasses only and do not involve any measurement.

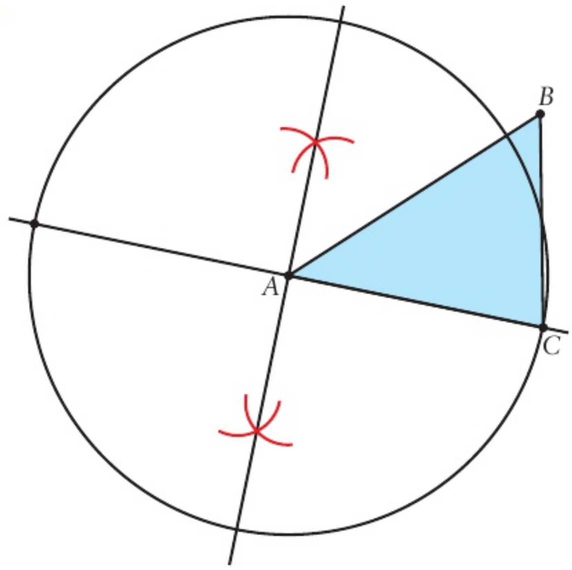
2 a



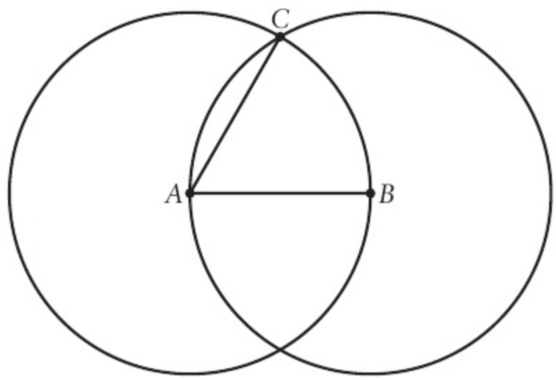
b



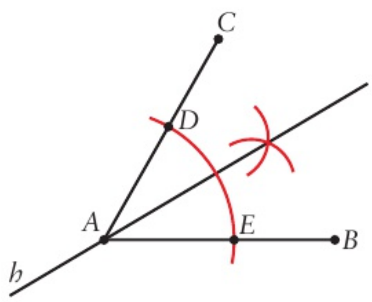
c



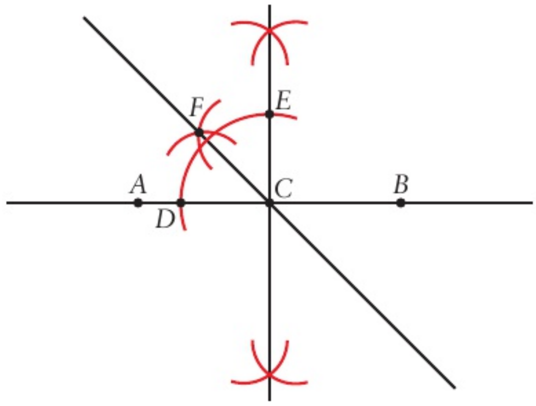
3 First, construct a 60° angle *CAB*



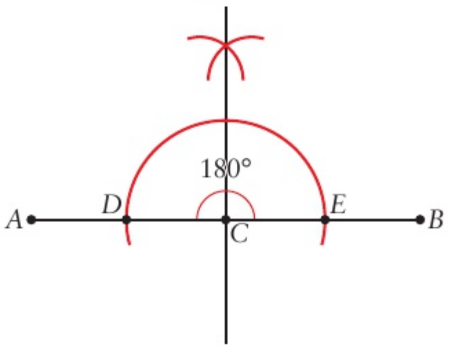
Then, bisect it.



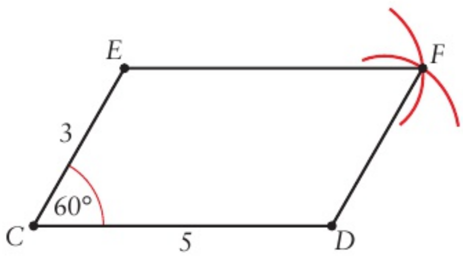
4 First, construct a perpendicular bisector of a segment AB. Then bisect the 90° angle ECA. The angle FCB is a 135° angle



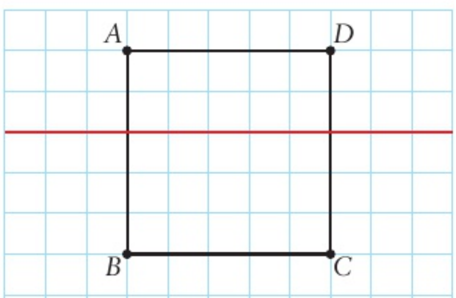
5 Bisect a 180° angle



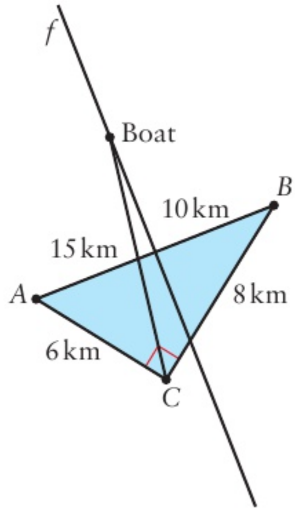
6



7 Let's consider side BC. The locus of points is the union of the two red parallel lines to BC.



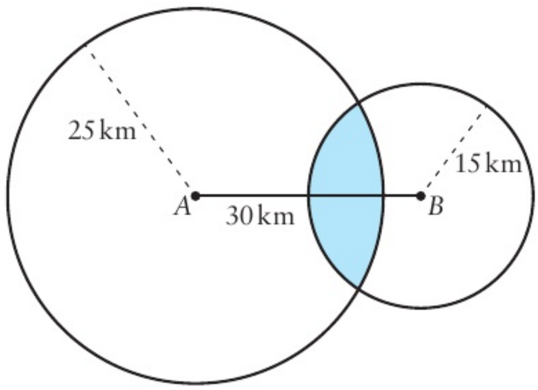
8 The points A, B and C form a right-angled triangle. The right angle should be marked on the diagram. Construct the perpendicular bisector f of AB. The boat is on the perpendicular bisector of AB and is 15 km from Town C. There are two points that are on the perpendicular bisector of AB at 15 km from C, but the boat is in the North Sea, so the boat is uniquely located and it is:



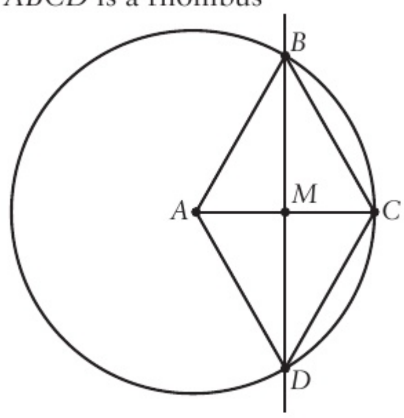
9 Equidistant from the endpoints of AB

10 Equidistant from both the arms of the angle

11



12 $ABCD$ is a rhombus



Chapter 9 answers

Do you recall?

1 a Shoe size is discrete data.

b Length of your foot is continuous data.

c Number of bananas is discrete data.

d The mass of a bunch of bananas is continuous data.

e The number of times you go to the store in a week is discrete data.

f The time you spend in the store is continuous data.

2 a Mean is 15.6 or $15\frac{5}{9}$, median is 13 and mode is 6

b Mean is 17, median is 18, and there are two modes: 18 and 21

3 Line graph, bar chart, pie chart.

4 a 24: stem is 2, leaf is 4. 57: stem is 5, leaf is 7. 129: stem is 12, leaf is 9. 12.4: stem is 12, leaf is 4. 2345: stem is 234, leaf is 5. 2.38: stem is 2.3, leaf is 8.

b

stem	leaf
0	3 3 4 7 8 8 9 9
1	0 0 1 2 4 5 6 9
2	1 2 2 3 4 7 9
3	0 2 5 6 6 6 9

key: 3|5 = 35

Practice questions 9.1

1 a Primary data

b Primary data

c Secondary data

2 a Which time slot do you usually use to come to the school canteen?

☐ 12:00 to 12:10 ☐ 12:10 to 12:20

☐ 12:20 to 12:30 ☐ 12:30 to 12:40

☐ 12:40 to 12:50

b Which days do you usually have lunch in the school canteen?

☐ Monday ☐ Tuesday

☐ Wednesday ☐ Thursday

☐ Friday

c To what extent do you agree with the statement: I like school lunches?

☐ Strongly agree ☐ Agree

☐ Neither agree nor disagree

☐ Disagree ☐ Strongly disagree

3 a Considering the traffic along our street, which one of these statements do you agree with most?

☐ The traffic speed is too fast all of the time.

☐ The traffic speed is too fast sometimes.

☐ The traffic speed is not too fast.

☐ Don't know.

b If the local authorities placed speed restrictions, what effect do you think it would have on the traffic?

☐ It would not make a difference to the speed of the traffic.

☐ The traffic would slow down.

☐ The traffic would speed up

☐ Don't know.

c How long have you lived on the street?

☐ less than 2 years

☐ 2 or greater and less than 5 years

☐ 5 or greater and less than 10 years

☐ 10 or greater and less than 20 years

☐ 20 years or more

4 a i Eastern Europe and Central Asia with 129

ii Least developed countries with 16

iii Eastern Europe and Central Asia, Western Europe, North America iv Sub-Saharan Africa, Eastern and Southern Africa, West and Central Africa, Least developed countries

b The Middle East and North Africa regions literacy rates are 91 for males and 88 for females. The world values are 92 and 85 respectively. For this region the literacy rates for males are very slightly below the world rate, but the literacy rate for females is slightly above the world rate.

5 Any online shopping site, e.g Amazon: the information related to book details, the author, the ISBN etc. School's database stores all the information regarding students' and staff's personal data, progress records, attendance records.

6 GDPR is an EU regulation that protects the personal data of individuals. It does not mean that personal data cannot be used. It means its use is protected. Personal data is information such as name, home address, email address and medical records. Data processing means collecting, recording, storing, sharing etc. For example it is against the EU GDPR regulations to record someone and share the recording. Although it is an EU regulation it also protects the personal data of EU citizens outside of the EU. https://gdpr-info.eu/ https://ec.europa.eu/info/law/law-topic/data-protection en
Not all countries have data protection laws and the level of protection of personal data varies across the world.

Practice questions 9.2.2

1 a Mean is 58.1, Median is 60.5, there is no modal value, the range is 89.
There is a large spread in the data, there is no mode and the median is larger than the mean.

b Mean is 41.1, Median is 52, mode is 1 and the range is 89.
There is a large spread in the data, the mode is much smaller than the mean and the median.

2 a Modal rainfall is 2, there are two sunshine modes 0 and 20

b Mean rainfall is 6.63, mean sunshine is 20.63

c Median rainfall is 2.5, median sunshine is 19.5

d Range of rainfall is 23, range of sunshine is 70

e To promote the town to tourists, the best measure of central tendency for rainfall would be the mode as it is the lowest average value. The best measure of sunshine would be the mean value as this is the highest average value.

f The sunshine value of 20.63 is fair as the mode of 20 and the median 19.5 are very close to the mean value. The rainfall value is a little less fair as the mean is 6.63 which is different to 2, however the median is also 2.5 which justifies the use of a value of 6.63. To be fairer to tourists the range should also be quoted as this gives them the worst and the best possible situations for their trip and they have more information to inform their decision.

3 a Mean for Australia is 0.964 mean for USA is 1.04

b Median for Australia is 0.91, median for USA is 0.73

c Range of Australia is $1.2 - 0.86 = 0.34$
Range of USA is $1.86 - 0.62 = 1.24$

d The mean of Australia data is lower than the mean of the USA data, but the median of the Australia data is higher than the median of the USA. Therefore, concluding which country has the higher interest rate depends on which average value you use. Both countries' interest rates have reduced over the period, however, as the USA median is much lower than the mean, it suggests the rate of decrease of the USA interest rates is greater than the Australian. This is supported by the range values. The USA has seen a larger change in their interest rates than Australia.

4 a Mean for Mexico is 96.9 and mean for Switzerland is 573

b Median for Mexico is 95.9, median for Switzerland is 562

c Range of Mexico is $109.8 - 91.3 = 18.50$
Range of Switzerland is $653.8 - 501.1 = 153$

d The mean and median of Mexico's data is much lower than the mean and median of Switzerland's data. Mexico's mean and median are quite close in value, however, Switzerland's mean is greater than its median suggesting the data values may be increasing each year. From these statistics we can infer that Switzerland's pharmaceutical sales per capita are much higher than Mexico's and they appear to be increasing. This is supported by the difference in the range values. Switzerland has a higher range value than Mexico suggesting the rate of increase in sales per capita is greater.

5 a Mean for Sweden is 44 367, mean for Latvia is 22 643

b Median for Sweden is 44 544.50, median for Latvia is 22 282.50

c Range of Sweden is $46\,695 - 41\,500 = 5195$
Range of Latvia is $28\,454 - 18\,316 = 10\,138$

d The mean and median of Latvia's data is much lower than the mean and median of Sweden's data, suggesting the salaries are higher in Sweden than in Latvia, but we don't know the cost of living in each country to suggest Sweden workers are in a better position. Latvia's mean is greater than the median. This suggests the data is increasing at a faster rate than in Sweden during the period 2010 to 2019. This is supported by the difference in the range values. The value for Latvia is greater than that for Sweden suggesting the rate of increase in salary is greater for Latvia.

6 a Mode for 2017 is 5

b Median for 2017 is 5

c Mean for 2017 is 4.62

d Range for 2017 is 6

7 a Mode for 2020 is 6

b Median for 2020 is 5

c Mean for 2020 is 5.07

d Range for 2020 is 5

e The mean and the mode for 2020 appears to show an increase in the IB scores compared to the 2017 classes. However, the median values are the same for 2017 and 2020, and the range is one less for 2020. This suggests that the 2020 classes are less spread out across the grades with fewer lower grades but also fewer level 7s. It is a valid point to include the nature of the exam system in this example. 2020 was the COVID year and grades were awarded using a different system rather than on examination performance.

Practice questions 9.2.3

1 a i Modal group is 11–20

ii median group is 21–30 (20.5 by gdc),

iii mean estimate is 24.6

b i Modal group is $10 < x \leq 20$

ii median group is $10 < x \leq 20$ (20 by gdc)

iii mean estimate is 20.7

2 **a** Mean cost is €6.54, median is €5, mode is €5

b Mean estimate is 5.38, median group is $0 < x \le 5$ (2.5 by gdc), modal group is $0 < x \le 5$

c Range for cost is $15 - 5 =$ €10

d The range for the distance travelled cannot be determined as we do not know where in the group $15 < x \le 20$ the maximum lies and we do not know where in the group $0 < x \le 5$ the minimum lies.

3 $a = 6$

4 Missing frequency value could be $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$

5 $a = 3$

6 $a = 5, b = 1$

Practice questions 9.3.1

1 Rainfall

0	0 1 2 2 3 4
1	8
2	3

Key: 2|1 = 21

Sunshine

0	0 0
1	5 9
2	0 0 1
3	
4	
5	
6	
7	0

Key: 2|1 = 21

These data sets are not suitable for back-to-back stem-and-leaf plots because they are not comparing like data. They are two different variables: rainfall and sunshine, and measured in different units, mm and hours.

2 **a**

Before		After
	1	8 9
	2	2 3
	3	4 8 8
9 4 1	4	1 4
6 5 4 4	5	1
5 3	6	
3	7	

Before key 2|3 = 32;
After key: 2|3 = 23

b Before data: mode is 54, median is 54.5, mean is 55.4, range is 32
After data: mode is 38, median is 36, mean is 32.8, range is 33

c The after data has a smaller mean, median and mode value implying that the treatment has reduced the level of organic residue. The spread of the data appears to be the same.

3 **a**

Physics		Chemistry
7	2	
6 5	3	9
8 8 5 4	4	4
8 6 4 1	5	8 9
1 0	6	1 7 8
9 2	7	1 3 3 3 6 9
	8	9
	9	1

Key: 2|7 = 27 for Chemistry
and 72 for Physics

b Physics data: mode is 48, median is 51, mean is 51.6, range is 52
Chemistry data: mode is 73, median is 71, mean is 681), range is 52

c The mean, median and mode of the chemistry data are all higher than the mean, median and mode of the physics data and the range is the same value for both. It implies that the students scored higher in the chemistry test, however ,we do not know if the students prepared more for the chemistry test or if the teaching was better in chemistry. So we can't support Bill's claim.

4 a Data set A: mode is 123, median is 130, mean is 130.7, range is 60
Data set B: mode is 169, median is 169, mean is168, range is 50

b Data set B has larger values than data set A as the mean, median and mode of data set B are all larger values than those of data set A. The data for set A appears to be more frequent at the upper end of the values.

c The data could represent the heights of students in a middle school class and a senior school class.

5

Day	Wed	Thurs	Fri	Sat	Sun
Freq	5	12	15	20	20

0	5
1	2 5
2	0 0

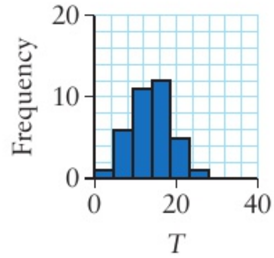
Key: 2|7 = 27

b The stem-and-leaf plot does not identify anything that the bar chart hasn't already highlighted. It is also a very small data set to use a stem-and-leaf plot for. It is too small to need sorting.

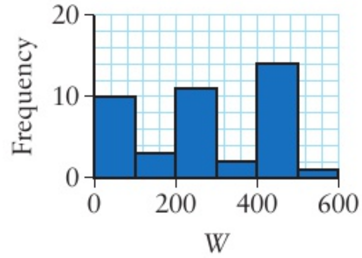
c If Simone did not know the range, she would not have been able to identify the Wednesday frequency, and this would have meant that she would need to use reasoning to deduce that there were 6 possible pairs of values for a and b. She would not have known which pair was the correct set of frequencies and she would have to have estimated from the chart's bar height ratios.

Practice questions 9.3.2

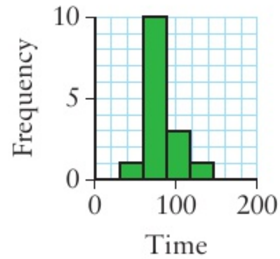
1 a



b



2



3 From the histograms, the modal group for both light bulbs is 260 to 280 hours. However, the Beamers have a higher frequency in this group, so there is more likelihood of the bulbs lasting this long for the Beamers. Looking closely, the Dazzlers have higher frequency in the 280 to 300 and the 300 to 320 groups and below 260 hours the two light bulbs appear to be the same. This implies that the Dazzlers are more reliable as they have more light bulbs lasting longer than 280 hours.

4 a The modal group for the control plants is $10 < h \le 15$; modal group for the experiment plants is $15 < h \le 20$

b Control frequency table

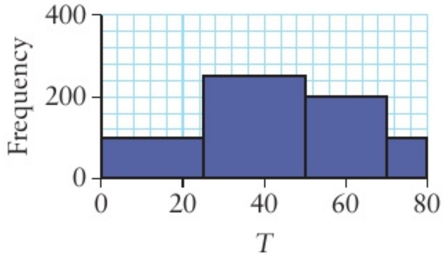
h	Frequency
$0 < h \le 5$	12
$5 < h \le 10$	30
$10 < h \le 15$	54
$15 < h \le 20$	24

Experiment frequency table

h	Frequency
$0 < h \le 5$	8
$5 < h \le 10$	12
$10 < h \le 15$	48
$15 < h \le 20$	52

c Median group for control plants is $10 < h \le 15$, median group for experiment plants is $10 < h \le 15$

d Mean estimate for control plants is 11.25 cm, mean estimate for experiment plants is 13.5 cm

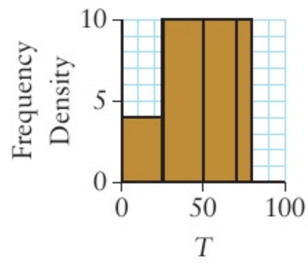
5 

b The data represents frequencies over different group widths. This leads to a misleading graph as a higher frequency can occur for a wider group naturally. A more representative chart would have the frequency as a proportion of the group width.

c Frequency density is a value that represents the frequency of the group relative to its width:

$$\text{frequency density} = \frac{\text{frequency}}{\text{group width}}$$

T	Frequency	Frequency density
$0 < T \le 25$	100	4
$25 < T \le 50$	250	10
$50 < T \le 70$	200	10
$70 < T \le 80$	100	10



d The histogram now gives a more representative view of the distribution of the frequencies and shows that there is an even spread from $25 < T \le 80$, rather than a misleading modal group at $25 < T \le 50$ just because the group width is 5 units wider than the next group and 15 units wider than the last group.

Check your knowledge questions

1 a continuous b discrete

c discrete d continuous

2 a Mean is 32.3, median is 31, mode is 48, range is 47

b Mean is 22.2, median is 18, mode is 12, range is 55

3 a Qu B as it gives options and is not open ended

b Qu B as it is not a leading question

c Qu A as it is more relevant

d Qu A as Qu B only makes sense to a Star Wars fan.

4 a Primary data because you can easily measure the heights of the students yourself.

b Secondary data because it would take too long to question all people and you could use the population census data.

c Secondary data as you would need to source this from a large organisation or various other sources.

d Primary data because you can measure the wildlife for yourself.

5 **a**

Mark	$20 \le M < 30$	$30 \le M < 40$	$40 \le M < 50$	$50 \le M < 60$	$60 \le M < 70$	$70 \le M < 80$	$80 \le M < 90$	$90 \le M < 100$
Freq	6	13	11	9	14	8	8	6

b Modal group is $60 \le M < 70$, median estimate using midpoint value is 55.5, (using ungrouped data or using the GDC the median is 57).

c Mean is 58.4 (If using the midpoint as an estimator for the group, the mean estimate would be 58.1.)

d Range is $98 - 21 = 77$

6

W	Frequency	Cumulative Frequency
$0 \le W < 100$	5	5
$100 \le W < 200$	13	18
$200 \le W < 300$	11	29
$300 \le W < 400$	14	43
$400 \le W < 500$	25	68
$500 \le W < 600$	10	78
$600 \le W < 700$	9	87
$700 \le W < 800$	5	92

Estimate of the median is the midpoint of the group $400 \le W < 500$, 450

7 **a**

Class A		Class B
8 1	2	
4	3	4 6 8 9
5	4	6 6
7 4	5	5 6
9 7 7 7	6	4 6 8
4	7	6 6
9 9	8	
8 1	9	1 2

Class A key : 5|6 = 65;
Class B key: 5|6 = 56

b Class A: range is 77, median is 67, mode is 67

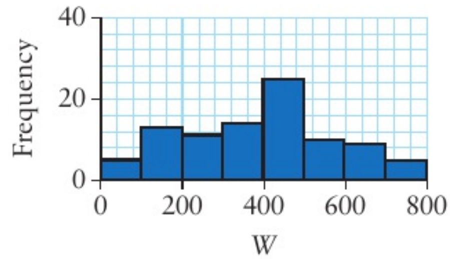
Class B: range is 58, median is 56, there are two modes: 46 and 76

c Class A mean is 63.3

Class B mean is 58.9

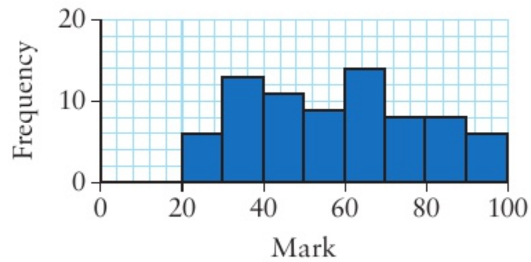
d The mean and median for class A are higher scores than class B, suggesting that class B scored better on the test. Class B has two modes one of which is higher than class A. This suggests that class B is divided into two sections. One section of the class did better than the other section. The range of scores in class A was more than the range of scores in class B. The observations are supported by the stem-and-leaf plot as class A scores appear more spread out in the diagram.

8



9

Mark	$20 \le M < 30$	$30 \le M < 40$	$40 \le M < 50$	$50 \le M < 60$	$60 \le M < 70$	$70 \le M < 80$	$80 \le M < 90$	$90 \le M < 100$
Freq	6	13	11	9	14	8	8	6



Modal group corresponds to the highest bar. The range was 77 and this corresponds to the chart as the bars are spread across the horizontal axes by 80 units. The median estimate was in the group which corresponds to the centre of the histogram and the mean value of 58.4 also corresponds to the centre of the chart.

10 **a**

Mark	$0 \le M < 10$	$10 \le M < 20$	$20 \le M < 30$	$30 \le M < 40$	$40 \le M < 50$	$50 \le M < 60$	$60 \le M < 70$
Freq	3	5	9	5	4	2	2

b **i** Modal group is $20 \le M < 30$

ii Median group is $20 \le M < 30$

iii Mean estimate is 30.3

c The modal group is clearly seen from the histogram and confirmed by the frequency table. The median group is more easily identified from the frequency table and using cumulative frequencies. The mean value can be determined from the histogram by identifying the midpoint of the group and the frequency from the vertical axis scale, however, using the frequency table helps to organise the calculations and communicate the process more clearly.

Chapter 10 answers

Do you recall?

1 **a** $\frac{1}{6}+\frac{1}{3}=\frac{1}{6}+\frac{2}{6}=\frac{3}{6}=\frac{1}{2}$

b $\frac{3}{4}-\frac{1}{2}=\frac{3}{4}-\frac{2}{4}=\frac{1}{4}$

c $\frac{1}{4}+\frac{2}{7}=\frac{7}{28}+\frac{8}{28}=\frac{15}{28}$

d $\frac{4}{5}-\frac{1}{2}=\frac{8}{10}-\frac{5}{10}=\frac{3}{10}$

2 **a** $0.92 + 0.03 = 0.95$

b $0.56 - 0.2 = 0.36$

c $0.14 + 0.5 = 0.64$

d $0.63 - 0.49 = 0.14$

3 **a** $\frac{1}{6}$ **b** $\frac{1}{2}$

4 **a** $\frac{3}{6}=\frac{1}{2}$ **b** $\frac{1}{2}$

Practice questions 10.1.1

1 **a** very unlikely **b** likely

c very likely **d** certain

2 **a** equally likely **b** unlikely

c likely **d** unlikely

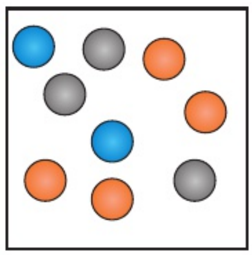
e impossible

3 **a** $\frac{1}{2}$ **b** $\frac{1}{3}$ **c** $\frac{2}{3}$ **d** $\frac{1}{6}$ **e** 1

4 **a** $\{1, 2, 3, 4, 5, 6\}$ **b** 6 options

c **i** $\frac{1}{2}$ **ii** $\frac{2}{3}$ **iii** $\frac{1}{2}$ **iv** $\frac{1}{2}$

5 **a**



b 9 marbles with 3 different options

c **i** $\frac{1}{3}$ **ii** $\frac{2}{9}$ **iii** $\frac{4}{9}$

6 **a** 14 chocolates with 3 different options

b white chocolate, $\frac{1}{2}$

c milk chocolate, $\frac{2}{14}=\frac{1}{7}$

7 **a** {Ariadne, Benji, Caroline, Daniel, Elissa}

b $\frac{1}{5}$ **c** $\frac{2}{5}$ **d** $\frac{4}{5}$

8 **a** $\frac{2}{3}$ **b** $\frac{3}{4}$ **c** 1

9 You roll a six-sided dice. However, unlike a standard 6-sided dice, this dice has no number 1, and instead shows the number 6 on two faces instead of just one.

a $\{2, 3, 4, 5, 6\}$

b Find the probability of rolling:

i $\frac{1}{6}$ **ii** $\frac{2}{6}=\frac{1}{3}$ **iii** $\frac{1}{2}$ **iv** $\frac{4}{6}=\frac{2}{3}$

10 a False, the likelihood of birth is not uniformly spread throughout the days of the year.

b False, it will depend on your ability with the new material.

c False, as names with Z are less frequent

d False, each die roll is independent, meaning it is not influenced by past rolls.

Practice questions 10.1.2

1 Sample space: $\{HH, HT, TH, TT\}$

a $\frac{1}{4}$ **b** $\frac{1}{4}$ **c** $\frac{1}{2}$

2 a $\frac{5}{36}$ **b** $\frac{3}{36} = \frac{1}{12}$

c $\frac{2}{36} = \frac{1}{18}$ **d** $\frac{1}{4}$

3 Sample space

1,1	1,2	1,3	1,4
2,1	2,2	2,3	2,4
3,1	3,2	3,3	3,4
4,1	4,2	4,3	4,4

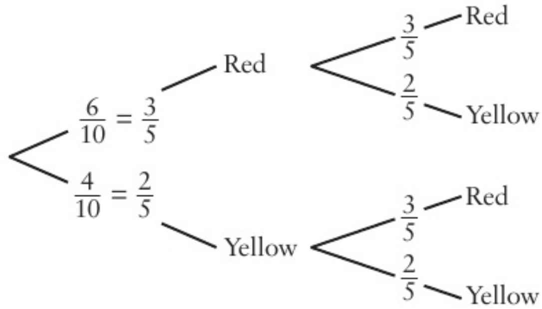
a $\frac{1}{4}$ **b** $\frac{1}{4}$ **c** $\frac{7}{16}$

4 a Sample Space

		1st draw			
		Hearts	Diamond	Spades	Clubs
2nd draw	Hearts	HH	DH	SH	CH
	Diamond	HD	DD	SD	CD
	Spades	HS	DS	SS	CS
	Clubs	HC	DC	SC	CC

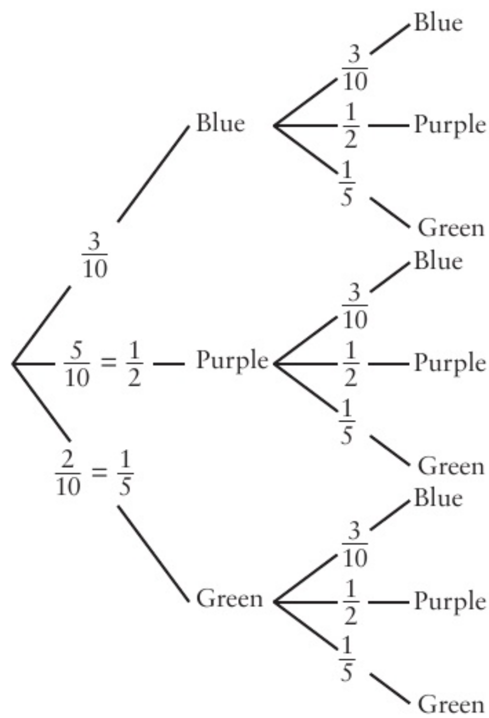
b $\frac{1}{16}$ **c** $\frac{1}{4}$ **d** $\frac{6}{16} = \frac{3}{8}$ **e** $\frac{7}{16}$

5 a 1st Draw 2nd Draw



b $\frac{9}{25}$ **c** $\frac{12}{25}$

6 a 1st draw 2nd draw



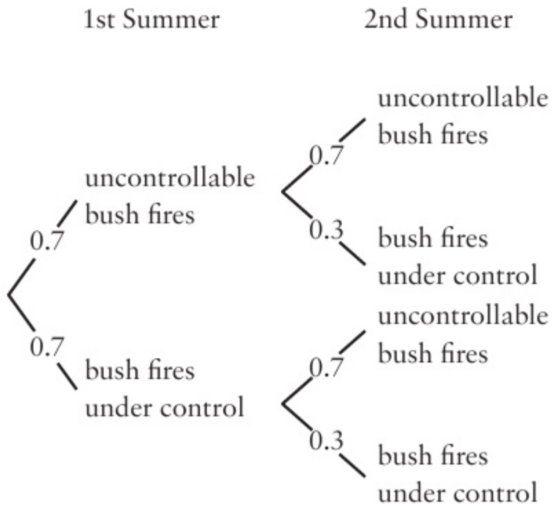
b i $\frac{1}{4}$ **ii** $\frac{1}{4}$

Practice questions 10.2

1 a $\frac{1}{100}$ **b** $\frac{1}{1000}$

c Example responses: global trends, increased flood defences, rainfall

2 a The bush fires are under control.

b 

c 0.49 d 0.343

e Example responses: It all depends on how accurate the estimate of 70% chance of a bush fire is. It also depends on the climate and whether we think this is constant.

3 a $\frac{18}{34} = \frac{9}{17}$ b $\frac{10}{34} = \frac{5}{17}$

c Example responses: no, as sport clothing tends to have more plastic

4 a i $\frac{2}{12} = \frac{1}{6}$ ii $\frac{6}{12} = \frac{1}{2}$ iii $\frac{6}{12} = \frac{1}{2}$

b Example responses: no, because the data may not be representative of other years, and global warming is causing more unpredictable weather conditions.

5 a i $\frac{4}{14} = \frac{2}{7}$ ii $\frac{6}{14} = \frac{3}{7}$

b No, because depending on the season, the amount of sun will vary.

6 a i $\frac{7}{52}$ ii $\frac{19}{52}$ b $\frac{5}{52}$

c Example responses: whether there is a lot of traffic in their area of the city, the sensitivity of their sensor

Check your knowledge questions

1 a unlikely b likely c (very) likely

2 a The probability of an even outcome is half.

b The probability of an outcome greater than 7 is 0.02

c The probability of a blue outcome is $\frac{8}{9}$

3 a Complementary events

b $\frac{2}{3}$ c $\frac{1}{3}$

4 a {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20}

b i $\frac{6}{20} = \frac{3}{10}$ ii $\frac{6}{20} = \frac{3}{10}$

iii $\frac{8}{20} = \frac{2}{5}$ iv $\frac{2}{20} = \frac{1}{10}$

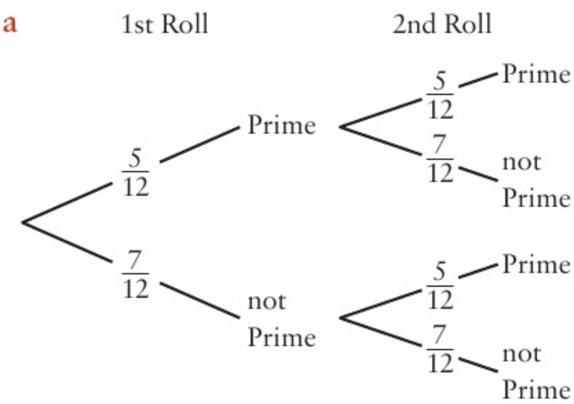
5 a 3, {yellow, green, orange}

b i $\frac{1}{2}$ ii $\frac{1}{3}$ iii $\frac{1}{6}$

6 a Sample Space

		Coin flip	
		Heads	Tails
Dice Roll	1	H1	T1
	2	H2	T2
	3	H3	T3
	4	H4	T4
	5	H5	T5
	6	H6	T6

b i $\frac{1}{2}$ ii $\frac{1}{2}$ iii $\frac{1}{4}$

7 a 

b $\frac{25}{144}$ c $\frac{35}{144}$ d $\frac{70}{144} = \frac{35}{72}$

Chapter 11 answers

Do you recall?

1 { },{1,3,5},{1},{3},{5},{1,3},{1,5},{3,5}

2 **b** and **e** are not part of Polya's pointers.

3 They are 500 000 cm or 5 km apart.

Practice questions 11.1

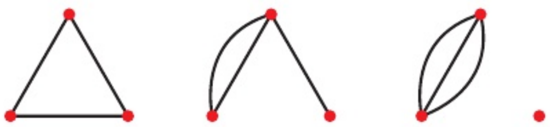
1 **I** 5 edges; 4 vertices; connected; simple

II 5 edges; 5 vertices; not connected; not simple

III 5 edges; 5 vertices; connected; not simple

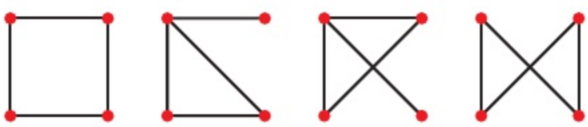
IV 4 edges; 5 vertices; not connected; simple

2 3 possible graphs:

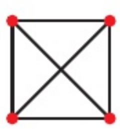


simple connected | connected not simple | not simple not connected

3 4 possible graphs:



4 Only 1 possible graph: K_4 it is the only possibility, as the graph is complete.



5 **a** Each vertex represents a metro station.

b The edges represent the tracks connecting the stations.

c 6 is the degree of Karlsplatz.

d There are 9 end-stations of degree 1, 9 crossings of degree 4 and 1 crossing of degree 5.

e They are at the end of a route.

f Any station has at least one line or edge. The map only shows stations.

6 **a** The metro map is simpler, which makes it easier to use.

b To improve the metro map, we could vary the distances to indicate the travel time.

7 **a** There are 10 vertices (destinations) in the map.

b A map that is to scale helps customers recognise where they are or where they want to go and the relative distances.

8 Stations where different lines meet are the busiest. Stations in the centre also tend to be very busy. Stations in the centre where lines meet are the busiest.

9 **a** Student's own answer or similar.

b Maps provide geographical features like proximity to coast. Maps also tend to be drawn to scale so show relative distances.

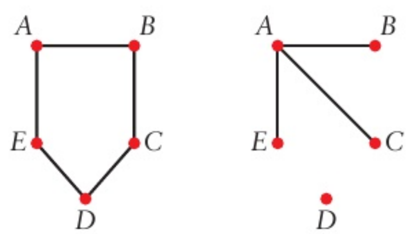
10 Research

Practice questions 11.2

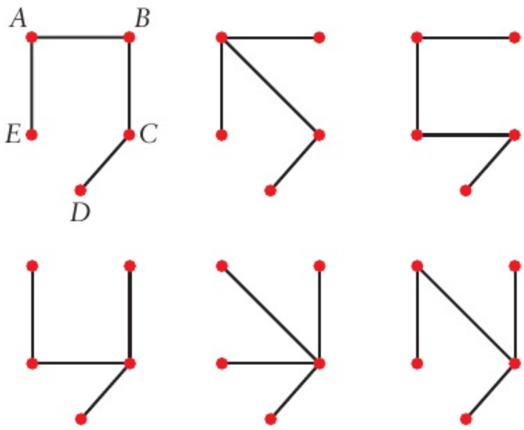
1 **a** No. **b** No.

2 Airlines are mostly organised as in Q.7 of 11.1, with one central logistic centre. Railways are more likely to be like a tree, taking you from one town to the next. However, in busier parts there are likely to be different connections between stations.

3 For example (many different possible answers):

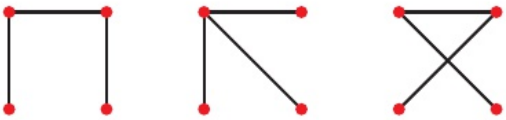


4 There are 12 possible spanning trees. Half of them are shown here. The other 6 are found by replacing edge CD by edge DE.



5 AD, BD, BE, CE

6



7 Ursus Major and Minor (Big and Small Dipper) is the easiest example. There may be others.

8 In a class of n students there will be $\frac{n(n-1)}{2}$ pairs.

With 4 students there are 6 pairs, with 5 students 10, with 6 students 15 pairs… The values are triangular numbers (1, 3, 6, 10, …).

Practice questions 11.3

1 a 3, 3, 2, 2 b 3, 3, 3, 3
c 2, 2, 2, 2, 2 d 4, 4, 4, 4, 4
e 3, 4, 4, 4, 4, 3 f 2, 3, 4, 4, 4, 3

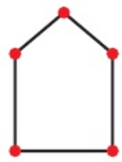
2 Eulerian circuit in c and d, Eulerian trail in a and f. There is no trail in e since the two vertices of odd degree are not adjacent.

3 The sum of the degrees is twice the number of edges, since each edge is counted twice in the degrees: once for its starting vertex and once for its end vertex.

Graph	Degrees	Edges
a	10	5
b	12	6
c	10	5
d	20	10
e	22	11
f	20	10

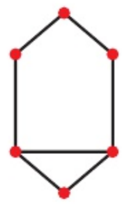
Graphs **b** and **d** are complete:

4 a

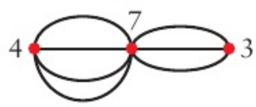


other solutions are possible

b



5 a Each room can be represented by a vertex and each door by an edge. We see that the room on the right has an odd number of doors and hence an odd degree to the vertex. Thus, a circuit is not possible.



b By adding or removing a door in the room to the right, all degrees will become even and a circuit will be possible.

6 Student's own floorplan. Most rooms only have one door, giving a vertex of odd degree. This makes a circuit impossible.

7 Student's own design.

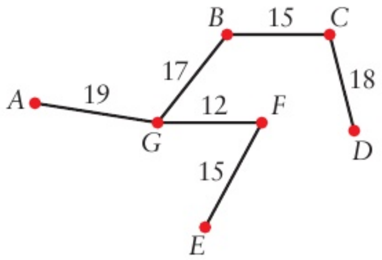
8 $7 \times 2 = 14$ is the minimum number of doors.

9 $2n$

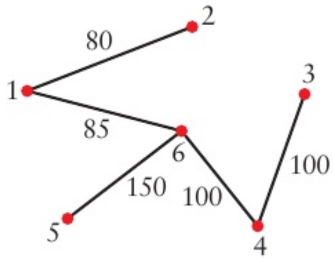
10 Complete graphs with an odd number of vertices have an Eulerian circuit, e.g. K_3, K etc. None of them have an Eulerian trail.

Practice questions 11.4

1

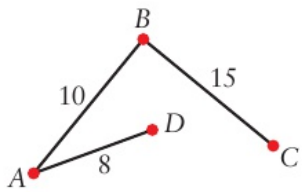


2



3 **a** $8 + 20 + 18 = 46$, so €46 000.

b See graph drawn with cost €33 000

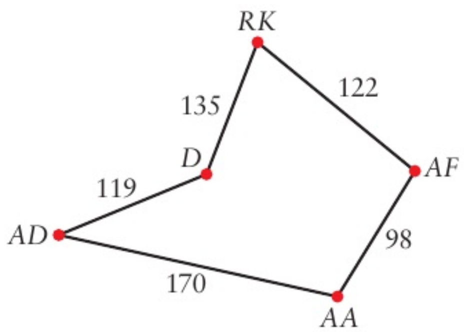


c €13 000 can be saved.

d W_4.

4 **a** Roads are rarely that straight over long distances; one of the roads is mostly on water.

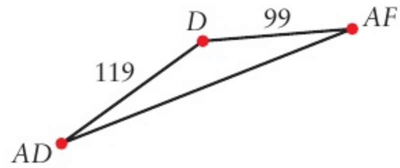
b



AD → D	119
D → RAK	135
RAK → AF	122
AF → AA	98
AA → AD	170
Total	644 km

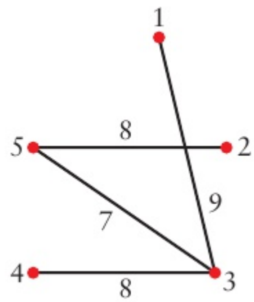
c Not every starting point will give the same answer.

d The requested distance will be less than $119 + 99 = 218$ km (see diagram). Hence an estimate of between 200–210 km seems realistic, assuming that the map is approximately to scale.



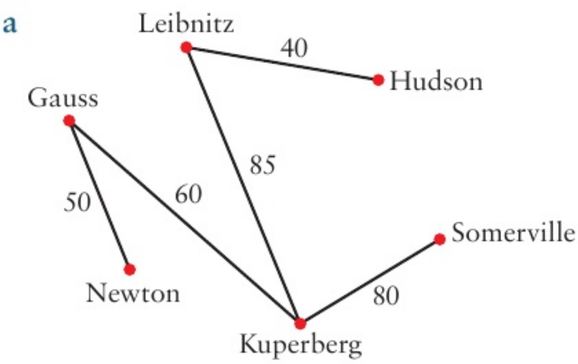
5 **a** £32 000 (see diagram)

b It depends on the accessibility of the different offices and how easy it is to lay the cable. This will depend on existing ducts, whether there is rock in the ground, etc.



c K_5 It is a complete graph.

6 **a**



b Kuperberg, most lines leave from there.

c Newton → Gauss → Kuperberg → Leibnitz → Hudson; it will take passengers 235 minutes, instead of 120.

Practice questions 11.5

1 Student's flow chart. Should contain waking up, dressing, etc. in a logical order.

2 a A, B, C, D, E
  b AB, AC, AD, BD, BE, CB, CE, DE
  c BE, BD–DE
  d A–C–E, A–C–B–E, A–C–B–D–E, A–B–E, A–B–D–E, A–D–E

3 a 6 vertices
  b 10 arcs
  c i A–B–C, A–B–E–C, A–B–E–F–D–C, A–B–E–F–D–E–C, A–F–D–E–C
    ii A–F–D, A–B–C–D, A–B–E–C–D, A–B–E–F–D. N.B. The pathways passing through E can contain the loop E–A–B–E.

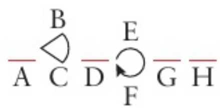
4 a The vertices represent intersections between different roads.
  b H–A–C–S or H–A–C–D–S. Both routes take 11 minutes.

5 a No circuit is possible, because there is no way to leave vertex D.
  b If we changed the direction of arc AD or CD this would be possible.
  c No, for the same reason as part **a**.

6 a The tour must end at E, as there is no way to get back to town from there.
  b A–D–C–B–F–E or A–B–F–D–C–E.
  c Yes, although the first one seems more logical, as in the second one you would pass by B and F twice.

7 A Heating the oven – this should be done when the dough has risen and runs in parallel with steps D–F
  B Making the pizza base (different possibilities, either bought readymade, or made with flour and water).
  C Optional: letting the dough rise when flour and water are used.
  D Getting the toppings ready.
  E Preparing/rolling the bottom.
  F Putting on the toppings.
  G Bake the pizza in the oven.

8 A Get permission to plant.
  B Decide on the precise location and date
  C Decide on the best kind of trees for your area.
  D Prepare the communication, inviting parent volunteers and informing about the benefits.



  E Send out the communication to the school community.
  F Make sure all members of the school community are informed (repeat communication if necessary), liaising with teachers.
  G Plant the trees.
  H Make sure there is a watering schedule for the trees in the summer months while they are saplings.

9 A Shape a committee composed of students, teachers, parents, school leadership and other staff interested.
  B Make an audit of the SUP used in the school.
  C Discuss alternatives with the relevant parties.
  D Communicate to the school community.

E Create an EcoCREW to allow students to lead and support the initiative.

F Get informed about the different kinds of plastic, its production and impact on environment and health.

G Inform the community about the initiative and what they can do at home to support it.

H Optional: join the PFC community, or another local group to share your story.

10 Very similar to the previous question. Can be taken on by the same EcoCREW.

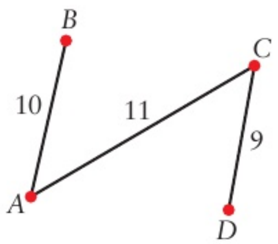
Check your knowledge questions

1 **a** 4 **b** 5 **c** 2, 3, 3, 2

d Yes, 2 vertices of odd degree e.g. BACDB

e $2 + 3 + 3 + 2 = 10 = 2 \times 5$. Each edge is counted twice.

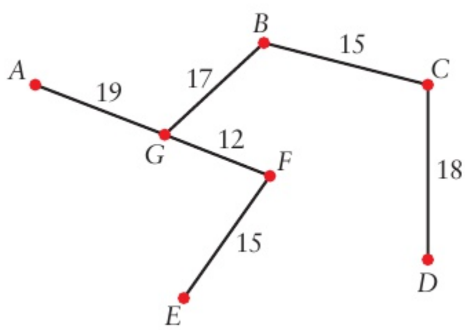
2 Weight: $9 + 10 + 11 = 30$



3 **a** A weighted graph.

b 5

c total weight: 96



4 **a** A directed graph.

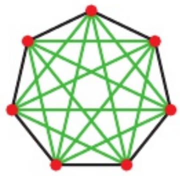
b **i** A–C–D **ii** B–A–C–D

c There is no way out of E.

d Not possible, can't leave E. CDBAEC requiring new edge EC. There may be other solutions.

5 **a** $\frac{7 \times 6}{2} = 21$.

b To check: 6 edges should connect to each of the other vertices.



6 $n - 1$.